Quadratics

A. QUADRATIC EQUATIONS

- A quadratic expression is any expression that can be rearranged to have the form $ax^2 + bx + c = 0$ where $a \neq 0$, and b and c can be any numbers, positive, negative, or zero

B. SOLVING QUADRATIC EQUATIONS

- How to solve quadratics equation:
  1. Factorization
  2. Completing the square
  3. Quadratic formula
  4. Drawing the graph
- Factorization

  Example:

  Factorize $x^2 - 40x - 6000 = 0$

  Solution:

  $$(x + 60)(x - 100) = 0$$
  $$x + 60 = 0 \rightarrow x = -60$$
  $$x - 100 = 0 \rightarrow x = 100$$

  The solution is $x = -60 \; or \; 100$
- Completing the square

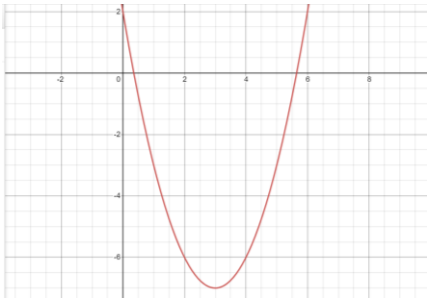
  Example:

  Solve the equation $x^2 - 6x + 2 = 0$ by completing the square

  Solution:

  $$x^2 - 6x = -2$$
  $$(x - 3)^2 = -2 + 9$$
  $$(x - 3)^2 = 7$$
  $$x - 3 = \pm\sqrt{7}$$
  $$x = 3 \pm \sqrt{7}$$

  This method can be used to find the vertex and the line of symmetry without plotting the points



Graph of $x^2 - 6x + 2 = 0$

  - The minimum point/vertex is $(3, -7)$
  - It has a line of symmetry, $x = 3$
- The quadratic formula

  To solve a general quadratic equation $ax^2 + bx + c = 0$ by completing the square:

  1. Divide both sides by a

  $$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

  2. Substract the constant term from both side

  $$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

  3. Take the coefficient of x

  $$+\frac{b}{a}$$

  4. Halve it

  $$+\frac{b}{2a}$$

  5. Square the answer

  $$+\frac{b^2}{4a^2}$$

  6. Add it to both sides of the equation

  $$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

  7. Factorise the left-hand side and tidy up the right-hand side

  $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

  8. Take the square root of both sides

  $$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  Example:

  Use the quadratic formula to solve $3x^2 - 6x + 2 = 0$

  Solution:

  $$a = 3, b = -6, c = 2$$

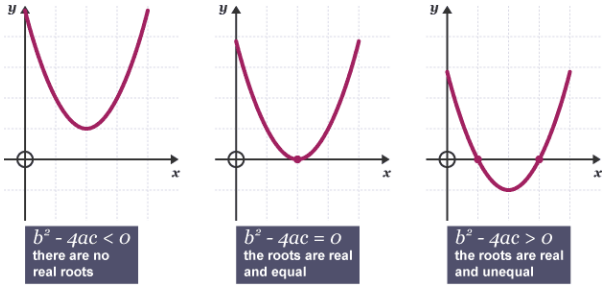
  Substitute these values to the formula

  $$x = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 2 \times 3}}{6}$$
  $$= \frac{6 \pm 2\sqrt{3}}{6}$$
  $$x = 0.423 \; or \; 1.577 \; (to \; 3 \; d.p.)$$

  The part under the square root sign determines whether the quadratic equation have real roots or not. This is called the discriminant

  $$D = b^2 - 4ac$$

  - If $b^2 - 4ac > 0$, the equation has two real roots
  - If $b^2 - 4ac = 0$, the equation has one repeated root
  - If $b^2 - 4ac < 0$, the equation has no real roots



C. SIMULTANEOUS EQUATIONS

- Simultaneous equations are two or more algebraic equations that share variables e.g. x and y
- There are 2 types of simultaneous equations, Linear simultaneous equations and Non-linear simultaneous equations
- How to solve linear simultaneous equations:
  1. Elimination
  2. Substitution
  3. Intersection of the graphs of the equations/equating
- Example:
  At a poultry farm, six hens and one duck cost $40, while four hens and three ducks cost $36. What is the cost of each type of bird?
  - Solution with elimination method
    Let the cost of one hen be $\$h$ and the cost of one duck be $\$d$, then
    $$6h + d = 40 \cdots eq.1$$
    $$4h + 3d = 36 \cdots eq.2$$
    $$eq.1 \times 3 \rightarrow 18h + 3d = 120$$
    Leave $eq.2$ as it is
    $$18h + 3d = 120$$
    $$4h + 3d = 36$$
    $$\underline{\qquad\qquad\qquad\qquad} \; -$$
    Subtracting $\rightarrow 14h = 84$
    $$h = 6$$
    Substitute $h = 6$ to $eq.1$
    $$36 + d = 40 \rightarrow d = 4$$
    Therefore, a hen costs $6 and a duck $4
  - Solution with substitution method
    Let the cost of one hen be $\$h$ and the cost of one duck be $\$d$, then
    $$6h + d = 40 \cdots eq.1$$
    $$4h + 3d = 36 \cdots eq.2$$
    Rearrange $eq.1$
    $$d = 40 - 6h$$
    Substitute to $eq.2$
    $$4h + 3(40 - 6h) = 36$$
    $$4h + 120 - 18h = 36$$
    $$14h = 84$$
    $$h = 6$$
    Substitute to $eq.1$
    $$d = 40 - 6 \times 6 = 4$$
    Therefore, a hen costs $6 and a duck $4
  - Solution with Intersection of the graphs of the equations method

Let $h = x$ and $d = y$
Red line is $4h + 3d = 36$
Blue line is $6h + d = 40$
From the graph, we can see that they intersect at $x = h = 6$ and $y = d = 4$
Therefore, a hen costs $6 and a duck $4

- Non-linear simultaneous equation is a system of two or more equations in two or more variables containing at least one equation that is not linear
- Non-linear simultaneous equation can be solved using the substitution method
  Example:
  Solve $x + 2y = 7$ and $x^2 + y^2 = 10$
  Solution:
  Rearrange $x + 2y = 7$
  $$x = 7 - 2y$$
  Substitute this to $x^2 + y^2 = 10$
  $$(7 - 2y)^2 + y^2 = 10$$
  $$49 - 28y + 4y^2 + y^2 - 10 = 0$$
  $$5y^2 - 28y + 39 = 0$$
  $$(5y - 13)(y - 3) = 0$$
  $$5y - 13 = 0 \rightarrow y = \frac{13}{5} = 2.6$$
  $$y - 3 = 0 \rightarrow y = 3$$
  Substitute to $x + 2y = 7$
  If $y = 2.6, x = 7 - \frac{26}{5} = 1.8$
  If $y = 3 \rightarrow x = 7 - 6 = 1$
  The solution is either $x = 1.8 \; and \; y = 2.6$
  Or $x = 1 \; and \; y = 3$
  Note:
  Always substitute into the linear equation. Substituting in the quadratic will give you extra answer which are not correct

D. QUADRATIC INEQUALITIES

- Example:
  Solve $x^2 - 4x + 3 > 0$
  Solution:
  $$(x - 1)(x - 3) > 0$$
  $$x = 1 \; or \; x = 3$$
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  1 3

Since we want the values of x for which $y > 0$, then the solution is

$$x < 1 \; or \; x > 3$$

- Example:

  Find the set values of k for which $x^2 + kx + 4 = 0$ has real roots

  Solution:

  A quadratic equation has real roots if $b^2 - 4ac \geq 0$, so the equation has real roots if

  $$k^2 - 4 \times 1 \times 4 \geq 0$$
  $$k^2 - 16 \geq 0$$
  $$k \geq 16$$

  So the set of values is $k \geq 4 \; and \; k \leq -4$