Trigonometry

A. TRIGONOMETRY BACKGROUND

- Angle of elevation: the angle between the horizontal and a direction above the horizontal
- Angle of depression: the angle between the horizontal and a direction below the horizontal
- Bearings or compass bearing: the direction measured as an angle from north, clockwise

B. TRIGONOMETRICAL FUNCTION

- $\sin\theta = \frac{opposite}{hypotenuse}$
- $\cos\theta = \frac{adjacent}{hypotenuse}$
- $\tan\theta = \frac{opposite}{adjacent}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- Special cases:
  1. The angles 30° and 60°
     - $\sin 30° = \frac{1}{2}$
     - $\cos 30° = \frac{\sqrt{3}}{2}$
     - $\tan 30° = \frac{1}{\sqrt{3}}$
     - $\sin 60° = \frac{\sqrt{3}}{2}$
     - $\cos 60° = \frac{1}{2}$
     - $\tan 60° = \sqrt{3}$
  2. The angle 45°
     - $\sin 45° = \frac{1}{\sqrt{2}}$
     - $\cos 45° = \frac{1}{\sqrt{2}}$
     - $\tan 45° = 1$
  3. The angles 0°-90°
     - $\sin 0° = 0$
     - $\cos 0° = 1$
     - $\tan 0° = 0$
     - $\sin 90° = 1$
     - $\cos 90° = 0$

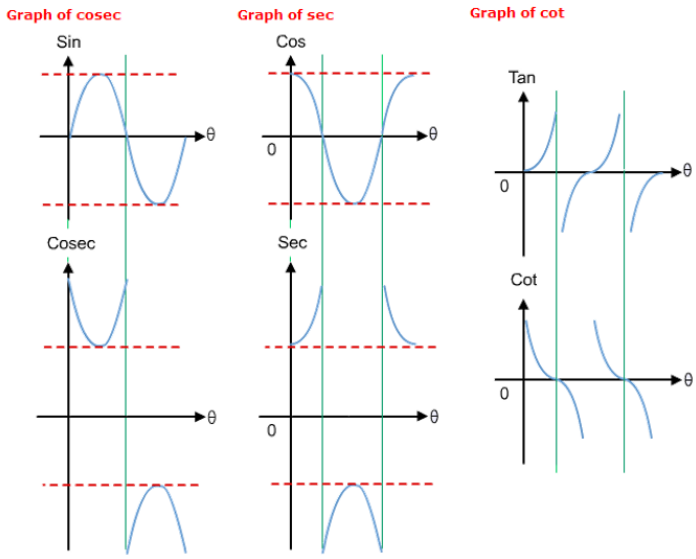
Positive and negative angles: Unless given in the form of bearings, angles are measured from the x-axis. Anticlockwise is taken to be positive and clockwise to be negative.

C. TRIGONOMETRICAL FUNCTIONS FOR ANGLES OF ANY SIZE

Identities:

- $(\cos\theta)^2 + (\sin\theta)^2$=1
- 1+$(\tan\theta)^2 = (\sec\theta)^2$
- $(\cot\theta)^2 + 1 = (\csc\theta)^2$

D. COSINE, SINE, AND TANGENT GRAPH



E. SOLVING EQUATIONS USING GRAPHS OF TRIGONOMETRICAL FUNCTIONS

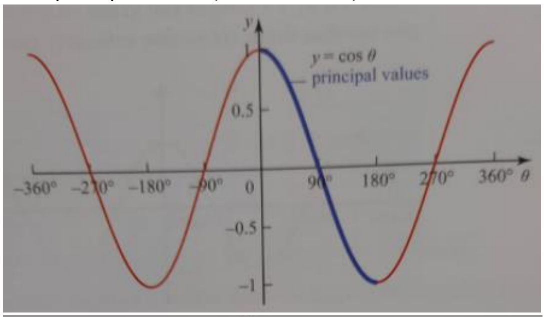
The functions cosine, sine, and tangent are all many-one mappings, so their inverse mapping are one-many. A functions has to be either one-one or many-one, so in order to define inverse functions for cosine, sine, and tangent, a restriction has to be placed on the domain of each so that it becomes a one-one mapping.
Restricted domains list.

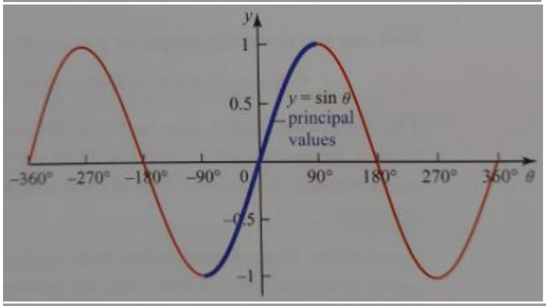
Function	Restricted domain (degrees)	Range
$f(\theta) = \sin\theta$	$-90° \le \theta \le 90°$	$-1 \le f(\theta) \le 1$
$g(\theta) = \cos\theta$	$0° \le \theta \le 180°$	$-1 \le g(\theta) \le 1$
$h(\theta) = \tan\theta$	$-90° \le \theta \le 90°$	$-\infty < h(\theta) < \infty$ All real numbers.

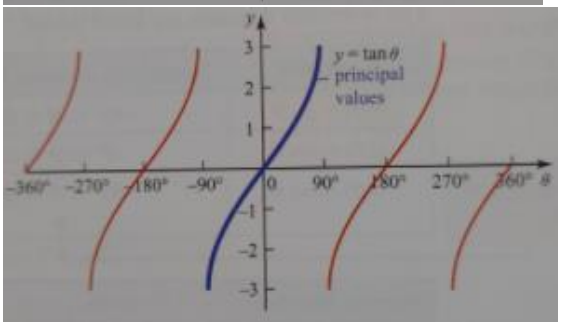
Inverse function	Domain	Range
$f^{-1}(\theta) = \sin^{-1}\theta$	$-1 \le \theta \le 1$	$-90° \le f^{-1}(\theta) \le 90°$
$g^{-1}(\theta) = \cos^{-1}\theta$	$-1 \le \theta \le 1$	$0° \le g^{-1}(\theta) \le 180°$
$h^{-1}(\theta) = \tan^{-1}\theta$	$-\infty < \theta < \infty$ All real numbers.	$-90° \le h^{-1}(\theta) \le 90°$

When you try to find $\cos^{-1} 0.5$ it will return just one answer, this value is called the **principal value** and it lies in the restricted domain.
To solve a trigonometric equation, you need to find all the roots in a given range.

The graphs of cosine, sine, and tangent together with their principal values (shows in blue).





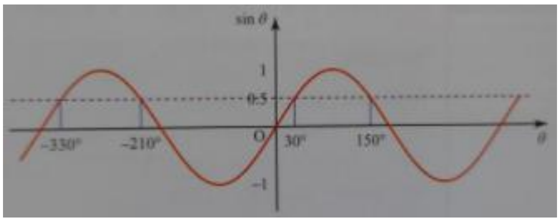


E.g

Find values of θ in the interval $-360° \leq \theta \leq 360°$ for which $\sin\theta = 0.5$

Solution :

$\sin\theta = 0.5 \Rightarrow \sin^{-1}\theta = 30°$. The graph of $\sin\theta$.



So, the values of θ for which $\sin\theta = 0.5$ are $-330°, -210°, 30°, 150°$.

F. CIRCULAR MEASURE

Some measure angles are degree, radian (rad), and grade (gra). The grade is a unit which was introduced to give a means of angle measurement which was compatible with the metric system. These are 100 grades in a right angle, so when you are in the grade mode, $\sin 100 = 1$, and $\sin 90 = 1$.

The radian is sometimes referred to as the natural unit of angular measure.

1 rad = 57.3°

Sometimes 1 radian written as 1°.

Since the circumference of a circle is given by $2\pi r$, it follows that the angle of a complete turn is 2π radians.

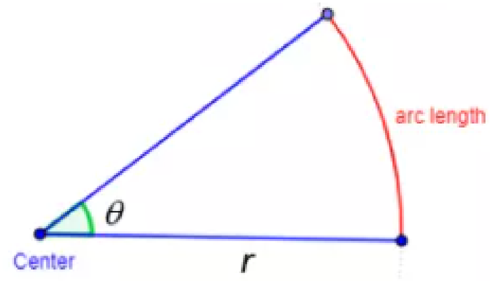
$360° = 2\pi$ radians.

$$1° = \frac{2\pi}{360} rad = 0.0175 \text{ rad}$$

- To convert degrees into radians, multiply by $\frac{\pi}{180}$
- To convert radians into degrees, multiply by $\frac{180}{\pi}$

G. THE LENGTH OF AN ARC OF A CIRCLE

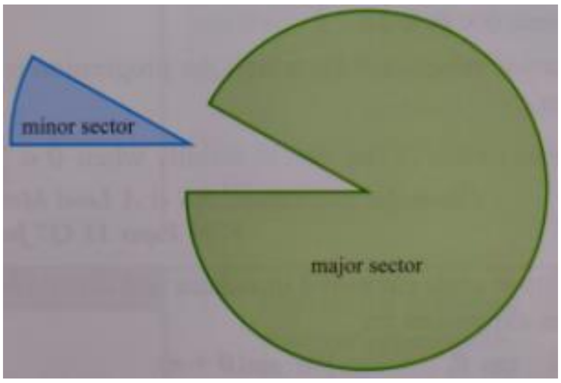
Arc length : $\frac{\theta}{2\pi}\cdot 2\pi r = r\cdot\theta$

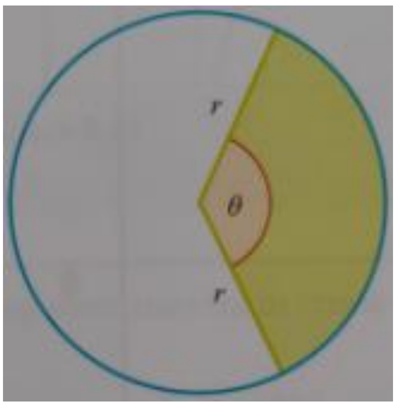


H. THE AREA OF A SECTOR OF A CIRCLE

Minor sector : the smaller sector than a semicircle.

Major sector : The larger sector than semicircle.





Area of sector : $\frac{\theta}{2\pi}\cdot \pi r^2 = \frac{1}{2}\theta r^2$

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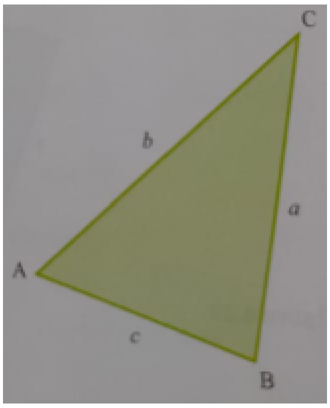
- The sine rule : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- The cosine rule : $a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \frac{b^2+c^2-a^2}{2bc}$
- The area of any triangle ABC = $\frac{1}{2}ab \sin C$.



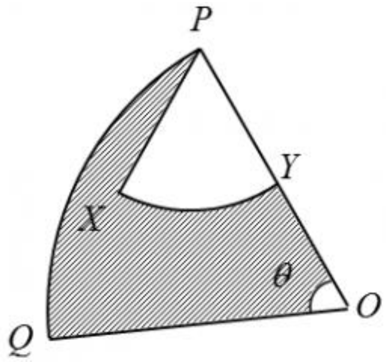
I. TRANSFORMATIONS AND GRAPHS OF TRIGONOMETRICAL FUNCTIONS

$$y = \pm A\ trig\ B(x \pm C) \pm D$$

- Amplitude
- The amplitude of a sinusoidal trig function (sine or cosine) is it's 'height,' the distance from the average value of the curve to its maximum (or minimum) value.
- $amplitude = |A|$
- The other trig functions (tangent, cotangent, secant, and cosecant) do not have an amplitude, but multiplication by A will affect their steepness. Note that a negative value of A will flip the graph of any function across the x-axis.
- Period
- The period of any trig function is the length of one cycle. $\sin\theta, \cos\theta, \sec\theta$, and $\csc\theta$ all have a period of 2π, while $\tan\theta$ and $\cot\theta$ have a period of π.
- $B = \frac{P}{|P'|}$
- P = original period
- P' = period in question/new period
- When $|P'|$ is larger than one, the new period is smaller than the original, so the function will appear horizontally compressed. When $|P'|$ is less than 1, the period is larger than the original, and the function will appear stretched.
- Phase shift
- $Phase\ shift = \frac{C}{B}$
  - $(+)$: the graph will shift to the right.
  - $(-)$: the graph will shift to the left.
- Vertical shift
  - $(+)$: the graph will shift up.
    $(-)$: the graph will shift down.

Exercise

1. Diagram below shows sector OPQ with center O and sector PXY with center P.



Given that $Q = 8\ cm, PY = 3\ cm, < XPY = 1.2\ rad$ and the length of arc $PQ = 6\ cm$, calculate

a) The value of θ, in radian.

b) The area, in cm^2, of the shaded region.

Solution :

a) $s = r\theta$
$6 = 8 \cdot \theta$
$\theta = 0.75$ rad

b) $Area\ of\ the\ shaded\ region$
$= Area\ of\ sector\ OPQ$
$- Area\ of\ sector\ PXY$
$= \frac{1}{2} \cdot (8)^2 \cdot (0.75) - \frac{1}{2} \cdot (3)^2 \cdot (1.2)$
$= 24 - 5.4$
$= 18.6\ cm^2$

2. Express $\frac{tan^2\theta - 1}{tan^2\theta + 1}$ in the form $asin^2\theta\ + b$, where a and b are constant to be found.

Solution :

$$\frac{\frac{sin^2\theta}{cos^2\theta} - 1}{\frac{sin^2\theta}{cos^2\theta} + 1} = \frac{\frac{sin^2\theta - cos^2\theta}{cos^2\theta}}{\frac{sin^2\theta + cos^2\theta}{cos^2\theta}}$$

$$= \frac{sin^2\theta - cos^2\theta}{sin^2\theta + cos^2\theta}$$

$$= \frac{sin^2\theta - (1 - sin^2\theta)}{1}$$

$$= 2sin^2\theta - 1$$