Sequences and Series

A. SEQUENCES

- Sequence is an ordered list (of numbers) with a *rule* relating the value of each term to its position in the list.

- A sequence could be denoted in the form
$$u_1, u_2, u_3, \dots, u_n$$
with natural (counting) numbers as subscripts, in increasing order from left to right. The n^{th} term, u_n, is usually called the *general term* of the sequence.

- A sequence could be defined as a function with natural numbers as its domain, i.e.
$$f(n), n \in \mathbb{Z}^+$$
The function relates each natural number *n* to the n^{th} term in the sequence.
e.g. For the sequence 2, 4, 8, 16, 32, 64, … ;
the rule is $f(n) = 2^n$

B. SERIES

- Series is the summation of the terms in a sequence.
- Series which continues indefinitely, usually denoted by … (3 dots) at the end, is called an *infinite* series.
e.g. $1 + 2 + 3 + \dots + n + \dots$

- Series which stops after a finite number of terms, usually denoted by a constant as the last term, is called a *finite* series.
e.g. $1 + 2 + 3 + 4 + 5 + 6$

C. THE SIGMA NOTATION

- The notation $\sum_{r=m}^{n} f(r)$ is shorthand for the summation
$f(m) + f(m+1) + f(m+2) + \dots + f(n)$

- The expression for the n^{th} term, u_n, could usually be deduced from the expression for the sum to (first) *n* terms, S_n, as follows:
$$u_n = S_n - S_{n-1}$$
$$\left[S_n \equiv \sum_{r=1}^{n} u_r = \sum_{r=1}^{n-1} u_r + u_n \equiv S_{n-1} + u_n\right]$$

D. ARITHMETIC SERIES

- An *Arithmetic Progression* is a sequence in which the difference between any term and its immediate previous term is a constant called the common difference.

- The n^{th} term, u_n, of an arithmetic progression is given by:
$$u_n = a + (n-1)d$$

- Series whose terms form an arithmetic progression is called an *Arithmetic Series.* The sum to (first) *n* terms, S_n, of an arithmetic series is given by:
$$S_n = \frac{1}{2}n[2a + (n-1)d] \text{ or } S_n = \frac{1}{2}[a + u_n]$$

E. GEOMETRIC SERIES

- A Geometric Progression is a sequence in which the ratio between any term and its immediate previous term is a constant called the common ratio.

- The n^{th} term, u_n, of the geometric progression is given by:
$$u_n = ar^{n-1}$$

- Series whose terms form a geometric progression is called a *Geometric Series.* The sum to (first) *n* terms, S_n, of a geometric series is given by:
$$S_n = \frac{a(1-r^n)}{1-r} \text{ or } S_n = \frac{a(r^n - 1)}{r-1}, r \neq 1$$

- Geometric series is *convergent* if the magnitude of the common ratio is less than one, i.e. $|r| < 1$, and the sum to infinity, S_∞, is given by:
$$S_\infty = \frac{a}{1-r}, |r| < 1$$

F. EXPANSION OF $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + nx^{n-1} + x^n$$

G. THE BINOMIAL THEOREM AND ITS APPLICATION

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

For $n \in \mathbb{Z}^+$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ And $0! = 1$

H. EXERCISE

1. [9709_s13_qp_11_004]

The third term of a geometric progression is -108 and the sixth term is 32. Find

(i) The common ratio
(ii) The first term
(iii) The sum to infinity

Answer:

(i) Third term $= ar^2 = -108$
Sixth term $= ar^5 = 32$

$$\frac{ar^5}{ar^2} = \frac{32}{-108} = -\frac{8}{27}$$

$$r^3 = -\frac{8}{27}$$

$$r = -\frac{2}{3}$$

(ii) $ar^2 = -108$

$$a\left(-\frac{2}{3}\right)^2 = -108$$

$$a\left(\frac{4}{9}\right) = -108$$

$$a = -243$$

(iii) Sum to infinite $\frac{a}{1-r}$

$$\frac{-243}{1-\left(-\frac{2}{3}\right)} = -\frac{729}{5} \text{ or } -145.8$$

2. [9709_s14_qp_11_005]

An arithmetic progression has first term a and common difference d. It is given that the sum of the first 200 terms is 4 times the sum of the first 100 times.

(i) Find d in terms of a.
(ii) Find the 100th term in terms of a.

Answer:

(i) $S_n = \frac{n}{2}(a + U_n)$

$$\frac{200}{2}(2a + 199d) = 4 \times \frac{100}{2}(2a + 99d)$$

$$2a + 199d = 4a + 198d$$

$$d = 2a$$

(ii) $U_{100} = a + 99d$

$$a + 99(2a) = 199a$$

3. [9709_s14_qp_11_003]

Find the term independent of x in the expansion of $\left(4x^3 + \frac{1}{2x}\right)^8$.

Answer:

$$C_6^8 \times (4x^3)^2 \times \left(\frac{1}{2x}\right)^6 = 28 \times 16x^6 \times \frac{1}{64x^6}$$

$$= 7$$