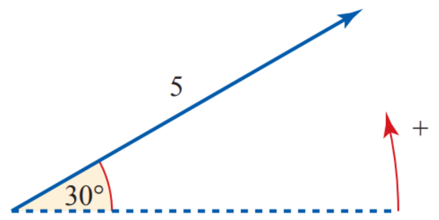
Vectors

A. VECTORS IN TWO DIMENSIONS

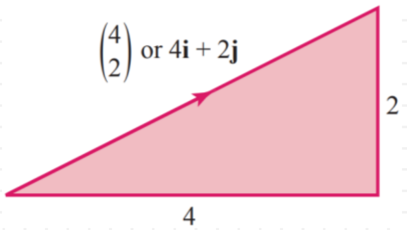
A quantity which has both size and direction is called a **vector**. Mass described by its size and no direction is associated with it, such a quantity is called a **scalar**.

In two dimensions, vector be depicted of a straight line with an arrowhead. **The length represents the size or magnitude of the vector and the direction is indicated by the line and the arrowhead.** Direction is usually given as the angle the vector makes with the positive x axis, with the anticlockwise direction taken to be positive.

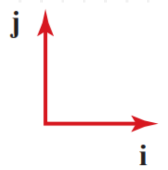


The vector in figure has magnitude 5, direction +30°. This is written (5, 30°) and said to be in magnitude–direction form or in polar form. The general form is written with (r, θ) **where r is its magnitude and θ its direction.**

The way of describing a vector is in terms of components and given directions. The vector in figure is 4 units in the x direction, and 2 in the y direction, and this is denoted by $\binom{4}{2}$.



This may also be written as 4i + 2j, where **i is a vector of magnitude 1, a unit vector in the x direction and j is a unit vector in the y direction.**

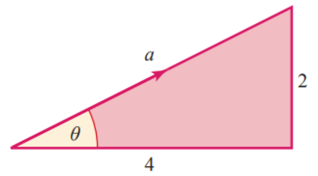


vector can be written as a line between two points with an arrow above it to indicate its direction, such as $\overrightarrow{OP}$. To convert a vector from component form to magnitude–direction form, or vice versa, is just a matter of applying **trigonometry to a right-angled triangle.**

E.g

Find the magnitude and direction of the vector a = 4i + 2j

Solution



The magnitude of a is given by the length a in figure

$a = \sqrt{4^2 + 2^2}$ → (using Pythagoras' theorem)

$= 4{,}47$ → (to 3 significant figures)

The direction is given by the angle θ.

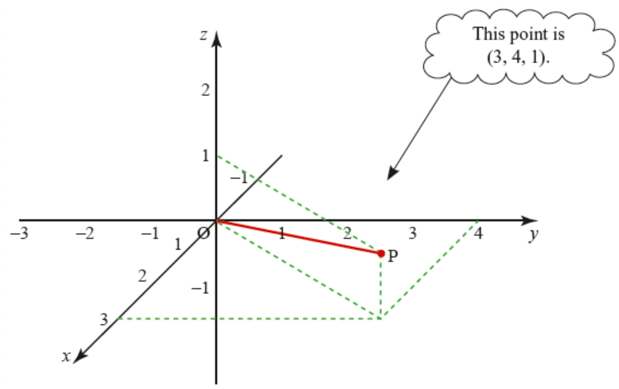
$\tan\theta = \frac{2}{4} = 0{,}5$

$\theta = 26{,}6°$ → (to 3 significant figures)

The vector a is (4.47, 26.6°). **The magnitude of a vector is also called its modulus and denoted by the symbols | |.** In the example a = 4i + 2j, **the modulus of a, written** $|\boldsymbol{a}|$, is 4.47. Another convention for writing the magnitude of a vector is to use the same letter, but in italics and not bold type; thus **the magnitude of a may be written a.**

B. VECTORS IN THREE DIMENSIONS

In three dimensions, a point **has three co-ordinates, usually called *x*, *y* and *z*.**



Where the point P is (3, 4, 1). The unit vectors ***i*, *j*** and ***k*** are used to describe vectors in three dimensions.

- **Equal vectors**

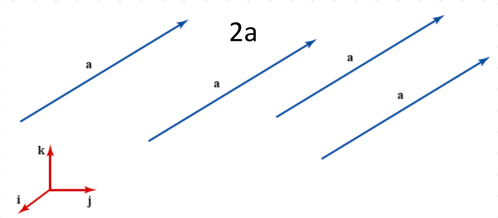
  The statement that two vectors **a** and **b** are equal means two things.

  1. **The direction of a is the same as the direction of b.**
  2. **The magnitude of a is the same as the magnitude of b.**

  If the vectors are given in component form, each component of **a** equals the corresponding component of **b**.

- **Position vectors**

  Saying the vector **a** is given by **3i** + **4j** + **k** tells you the components of the vector, or equivalently its magnitude and direction.
  All of the lines in figure represent the vector **a**.

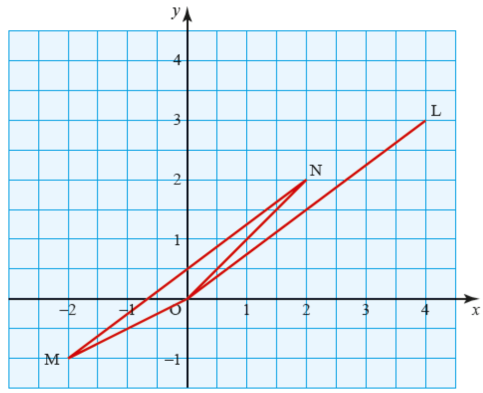


vector which starts at the origin is **position vector.**

Displacement vectors

Vectors can also be used to represent displacement. Note that magnitude of displacement is not the same as distance.

E.g



- The positions vector of L is $\overrightarrow{OL}, \overrightarrow{OL}$ and vector $\overrightarrow{MN} = \binom{4}{3}$.
- Since $\overrightarrow{OL} = \overrightarrow{MN}$, lines OL and MN are parallel and equal in length.
  A line joining two points, like MN is called a **line segment.** Also vector $\overrightarrow{MN}$ is an example of a **displacement vector.**

The length of a vectors

In two dimensions, the us of Pythagoras Theorem leads to the result that a vector $a_1 i + a_2 j$ has length $|a|$given by

$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2}$.

The length of the three-dimensional vector $a_1 i + a_2 j + a_3 k$ given by

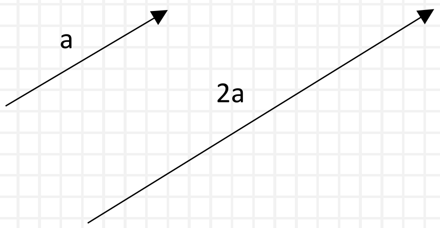
$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

C. VECTORS CALCULATIONS

Multiplying a vectors by a scalar

In vector multiplication with a scalar, its length is altered but its direction remains the same.

E.g



The vectors 2a is twice as long as the vector a but in the same direction. When the vector is in component form, each component is multiplied by the number.

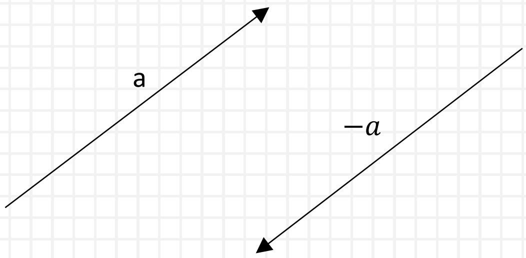
E.g

$$2 \times (4i + j - 3k) = 8i + 2j - 6k$$

$$2 \times \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}$$

The negative of a vector

The vector $-a$ has the same length as the vector a but the opposite direction.



In the component form, the components of $-a$ are the same as these for a but with their signs reversed. So

$$-\begin{pmatrix} 15 \\ -9 \\ -12 \end{pmatrix} = \begin{pmatrix} -15 \\ 9 \\ 12 \end{pmatrix}$$

Adding vector

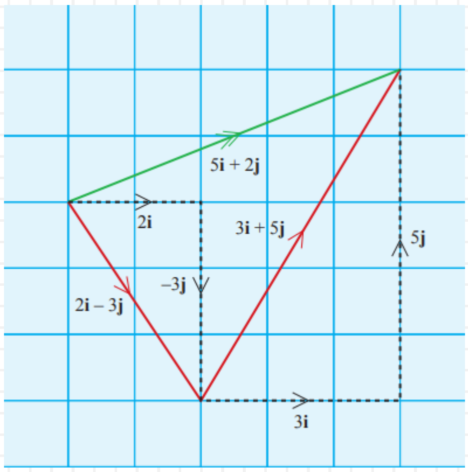
When vectors are given in the component form, they can be added component by component. This process can be seen geometrically by drawing them on graph paper.

E.g

Add the vectors $2i - 3j$ and $3i + 5j$

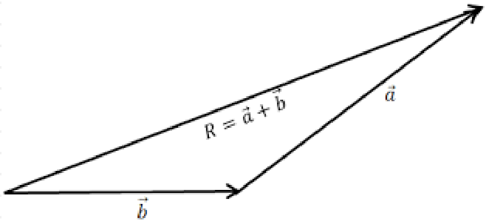
Solution

$$2i - 3j + 3i + 5j = 5i + 2j$$



The sum of two or more vectors is called the resultant and is usually indicated by being marked with two arrowheads.

When vectors are given in the magnitude–direction form, you can find their resultant by making a scale drawing, as in the example below



Subtracting vector

If we want the subtraction of two vectors, A and B is expressed mathematically as:

$R = A - B$ or $R = A + (-B)$

The vectors B and –B will have the same magnitude, but –B's direction will be opposite to that of vector B.

E.g

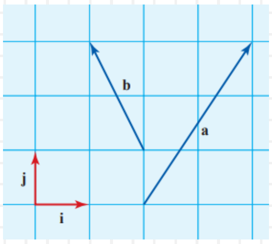
Two vectors, $a = 2i + 3j$ and $b = -i + 2j$.
Find $a - b$ and draw diagrams showing a, b, $a - b$.

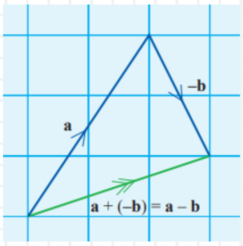
Solution

- Find $a - b$

$$a - b = (2i + 3j) - (-i + 2j) = 3i + j$$

- Draw diagrams showing a, b, $a - b$.





Geometrical figures

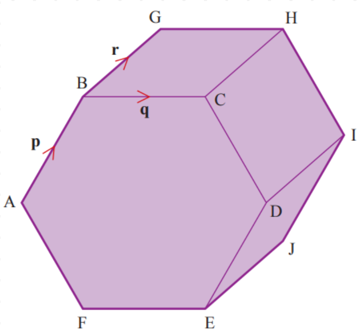
It is often useful to be able to express lines in a geometrical figure in terms of given vectors.

E.g

The hexagonal cross-section is regular and consequently $\overrightarrow{AD} = 2\overrightarrow{BC}$. Express the following in terms of **p**, **q**, and **r**.

a) $\overrightarrow{AC}$

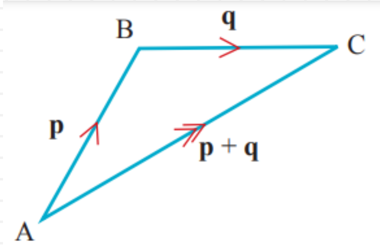
b) $\overrightarrow{AD}$



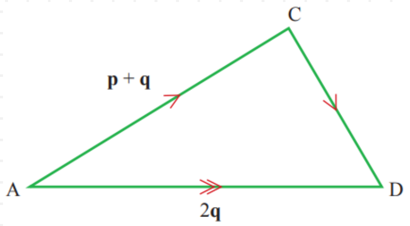
Solution

a) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AC} = \boldsymbol{p} + \boldsymbol{q}$



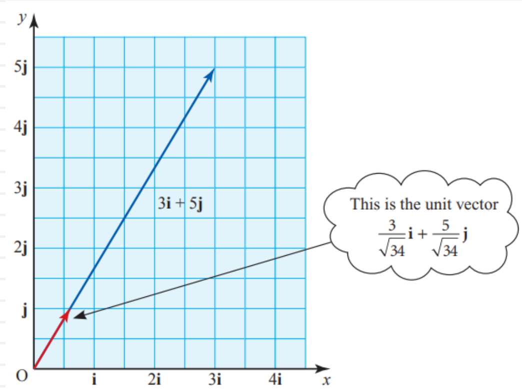
b) $\overrightarrow{AD} = 2\overrightarrow{BC} = 2\boldsymbol{q}$



Unit vectors

Unit vectors are vectors whose magnitude is exactly 1 unit, like **i** and **j**. To find the unit vector in the same direction as a given vector, divide that vector by its magnitude.

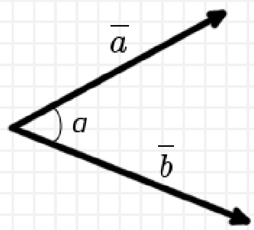
The vector $3i + 5j$ has magnitude $\sqrt{3^2 + 5^2} = \sqrt{34}$, so the vector $\frac{3}{\sqrt{34}}i + \frac{5}{\sqrt{34}}j$ is a unit vector. It has magnitude 1.



The unit vector in the direction of vector a is written $\hat{a}$ and read as 'a hat'.

D. THE ANGLE BETWEEN TWO VECTORS

Vectors are oriented in different directions while forming different angles. This angle exists between two vectors and is responsible for specifying the erection of vectors.



The angle between two vectors can be found using vector multiplication. There are two types of vector multiplication, i.e., scalar product and cross product. We know the dot product :

$$a \cdot b = |a||b| \cdot \cos\theta$$

$|a|$ is the length of a, and $|b|$ is the length of b, and θ is the angle between them. Now, the angle between two vectors formula is :

$$\theta = \cos^{-1}\frac{a \cdot b}{|a||b|}$$

$$\theta = \cos^{-1}\frac{a_1b_1+a_2b_2}{\sqrt{a_1{}^2+a_2{}^2}\times\sqrt{b_1{}^2+b_2{}^2}}$$

For angle between two vectors in 3D

$$\theta = \cos^{-1}\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}\times\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}}$$

E.g

Find the angle between the vectors $a = \begin{pmatrix}4\\-2\\3\end{pmatrix}$ and $b = \begin{pmatrix}1\\3\\-1\end{pmatrix}$.

Solution

We'll start by finding the dot product of a and b, $a \cdot b$

$a \cdot b = (4)(1) + (-2)(3) + (3)(-1)$

$a \cdot b = 4 - 6 - 3$

$a \cdot b = -5$

Now we'll use the distance formula to find the length of each vector, remembering that the initial point of both vectors is the origin

$|a| = D_a = \sqrt{(4)^2 + (-2)^2 + (3)^2}$

$|a| = D_a = \sqrt{16 + 4 + 9}$

$|a| = D_a = \sqrt{29}$

$|b| = D_b = \sqrt{(1)^2 + (3)^2 + (-1)^2}$

$|b| = D_b = \sqrt{1 + 9 + 1}$

$|b| = D_b = \sqrt{11}$

Plugging everything we've calculated into our formula for the angle between two vectors, we get

$\theta = \cos^{-1}\frac{a \cdot b}{|a||b|}$

$\theta = \cos^{-1}\frac{-5}{|\sqrt{29}||\sqrt{11}|}$

$\theta = \cos^{-1}\frac{-5}{|\sqrt{319}|}$

$\theta = 106.3°$

- **Perpendicular vectors**
  Since $\cos 90° = 0$, it follows that if vectors a and b are perpendicular then a .b = 0. Conversely, if the scalar product of two non-zero vectors is zero, they are perpendicular.
  E.g
  Show that the vectors $a = \begin{pmatrix}2\\4\end{pmatrix}$ and $b = \begin{pmatrix}6\\-3\end{pmatrix}$ are perpendicular .

  Solution
  The scalar product of the vectors is
  $a \cdot b = \begin{pmatrix}2\\4\end{pmatrix} \cdot \begin{pmatrix}6\\-3\end{pmatrix}$
  $a \cdot b = 2 \times 6 + 4 \times (-3)$
  $a \cdot b = 12 - 12 = 0$

  Therefore the vectors are perpendicular.
- **Further points concerning the scalar product**
  Scalar product of two vectors is an ordinary number. It has size but no direction and so is a scalar, rather than a vector. There is another way of multiplying vectors that gives a vector as the answer; it is called the **vector product.** The scalar product is calculated in the same way for three-dimensional vectors. In general :
  $$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \cdot \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$
  The scalar product of two vectors is commutative. It has the same value whichever of them is on the left-hand side or right-hand side. Thus $a \cdot b = b \cdot a$

EXERCISE

1. Find the magnitude of these vector 2i + 4j + 2k.

   Solution :
   2i + 4j + 2k can be written in the form vector
   $\boldsymbol{a} = \begin{pmatrix}2\\4\\2\end{pmatrix}$.
   $|a| = \sqrt{2^2 + 4^2 + 2^2}$
   $= \sqrt{4 + 16 + 4}$
   $= \sqrt{24}$
   $= 4{,}9$

2. Find unit vector in the direction of vector $\vec{a} = 2i + 3j + k$.

   Solution :
   Given $\vec{a} = 2i + 3j + k$

Magnitude of $|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2}$
$|\vec{a}| = \sqrt{4 + 9 + 1}$
$= \sqrt{14}$

Unit vector in direction $= \frac{1}{magnitude\ of\ \vec{a}} x\vec{a}$
$= \frac{1}{\sqrt{14}}(2i + 3j + 1k)$
$= \frac{2}{\sqrt{14}}i + \frac{3}{\sqrt{14}}j + \frac{1}{\sqrt{14}}k$

3. Find the angle between two vectors
$\vec{A} = 2i + 3j - 4k$ and $\vec{B} = 5i + 2j + 4k$.

Solution :
Consider the vector
$\vec{A} = 2i + 3j - 4k$ and $\vec{B} = 5i + 2j + 4k$
Now, as we know $\vec{A} . \vec{B} = A\ B \cos\theta$
θ is the angle between $\vec{A}$ and $\vec{B}$.
Therefore :

$$\cos\theta = \frac{\vec{A} . \vec{B}}{A\ .\ B}$$

$$\cos\theta = \frac{(2i + 3j - 4k)\ .\ (5i + 2j + 4k)}{\sqrt{2^2 + 3^2 + (-4)^2}\ .\ \sqrt{5^2 + 2^2 + 4^2}}$$

$$= \frac{(10 + 6 - 16)}{\sqrt{2^2 + 3^2 + (-4)^2}\ .\ \sqrt{5^2 + 2^2 + 4^2}} = 0$$

Therefore, $\cos\theta = 0$
$0 = 90°$
Hence angle between the vectors is 90°.