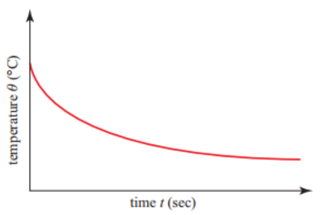
Differential Equation

Suppose you are in a hurry to go out and want to drink a cup of hot tea before you go.

How long will you have to wait until it is cool enough to drink?

To solve this problem, you need to know about the rate at which liquids cool at different temperatures.



The rate of change of temperature is numerically greatest at high temperature and gets numerically less as the temperature drops. The rate of change is always negative since the temperature is decreasing.

The gradient of the temperature graph may be written as $\frac{d\theta}{dt}$ where θ is the temperature of the liquid and t is the time. The quantitiy $\frac{d\theta}{dt}$ tells us the rate at which the temperature is increasing. As the liquid is cooling, $\frac{d\theta}{dt}$ will be negative, the rate of cooling may be written as $-\frac{d\theta}{dt}$.

The difference in temperature of the liquid and that of the surrounding air may be written as $\theta - \theta_0$, where θ_0 is the temperature of the surrounding air and can be expressed mathematically as:

$-\frac{d\theta}{dt} \propto (\theta - \theta_0)$ or $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

Where k is a positive constant.

Any equation which involves a derivative such as, $\frac{d\theta}{dt}, \frac{dy}{dx}, or\ \frac{d^2y}{dx^2}$, is known as a *differential equation*.

A differential equation which only involves a first derivatives such as $\frac{dy}{dx}$ is called *first-order differential equation*, $\frac{d^2x}{dy^2}$ is called *second-order differential equation* and so on.

FORMING DIFFERENTIAL EQUATIONS FROM RATES OF CHANGE

If you are given sufficient information about the rate of change of a quantity, such as temperature or velocity, you can work out a differential equation to model the situation.

For example, if the altitude of an aircraft is being considered, the phrase 'the rate of change of height' might be used. This actually means 'the rate of change of height with respect to time' and could be written as $\frac{dh}{dt}$. If the height of aircraft changes according to the horizontal distance, you would talk about 'the rate of change of height with respect to time' and could write this as $\frac{dh}{dx}$, where x is the horizontal distance travelled.

EXERCISE

1. [9709_s16_qp_31_04]

The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = y(1-2x^2),$$

And it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms.

Answer:

$x\frac{dy}{dx} = y(1-2x^2)$

$\frac{1}{y}dy = \frac{1-2x^2}{x}dx$

$\int \frac{1}{y}dy = \int \frac{1-2x^2}{x}dx$

$\int \frac{1}{y}dy = \int \frac{1}{x} - 2x\,dx$

$\ln y = \ln x - x^2 + c$

For $y = 2, x = 1$

$\ln 2 = \ln 1 - 1 + c$

$c = 1 + \ln 2$

$\ln y = \ln x - x^2 + 1 + \ln 2$

$\ln y = \ln 2x - x^2 + 1$

$y = 2x + e^{1-x^2}$

2. [9709_w17_qp_31_06]

The variables x and y satisfy the differential equation

$\frac{dy}{dx} = 4\cos^2 y \tan x$

For $0 \le x < \frac{1}{2}\pi$, and $x = 0$ when $y = \frac{1}{4}\pi$. Solve this differential equation and find the value of x when $y = \frac{1}{3}\pi$.

Answer:

$\frac{dy}{dx} = 4\cos^2 y \tan x$

$\frac{1}{\cos^2 y}dy = 4\tan x\,dx$

$\int \sec^2 y\,dy = \int 4\tan x\,dx$

$\tan y = -4\ln\cos x + c$

When $x = 0, y = \frac{\pi}{4}$

$\tan\frac{\pi}{4} = -4\ln\cos 0 + c$

$c = \tan\frac{\pi}{4} = 1$

$\tan y = -4\ln\cos x + 1$

When $y = \frac{1}{3}\pi$

$\tan\frac{\pi}{3} = -4\ln\cos x + 1$

$\tan\frac{\pi}{3} - 1 = -4\ln\cos x$

$\frac{1}{4}\left(1 - \tan\frac{\pi}{3}\right) = \ln\cos x$

$x = \cos^{-1}\left(e^{\frac{1}{4}\left(1-\tan\frac{\pi}{3}\right)}\right)$

$x = 0.587$

3. **[9709_s18_qp_31_06]**
In a certain chemical reaction the amount, x grams, of a substance is decreasing. The differential equation relating x and t, the time in seconds since the reaction started, is

$$\frac{dx}{dt} = -kx\sqrt{t}$$

Where k is a positive constant. It is given that $x = 100$ at the start of the reaction.

(i) Solve the differential equation, obtaining relation between x, t, and k.

(ii) Given that $t = 25$ when $x = 80$, find the value of t when $x = 40$

Answer:

(i) $\frac{dx}{dt} = -kx\sqrt{t}$

$\int \frac{1}{x}\,dx = \int -k\sqrt{t}\,dt$

$\ln x + c = -\frac{2}{3}kt^{\frac{3}{2}}$

$x = 100, t = 0$

$\ln 100 + c = -\frac{2}{3}k0^{\frac{3}{2}}$

$c = -\ln 100$

$\ln x - \ln 100 = -\frac{2}{3}kt^{\frac{3}{2}}$

(ii) $t = 25, x = 80$

$\ln x - \ln 100 = -\frac{2}{3}kt^{\frac{3}{2}}$

$\ln 80 - \ln 100 = -\frac{2}{3}\left(25^{\frac{3}{2}}\right)k$

$k = -\frac{3}{250}\ln\frac{4}{5}$

$\ln x - \ln 100 = -\frac{2}{3}\left(-\frac{3}{250}\ln\frac{4}{5}\right)t^{\frac{3}{2}}$

$x = 40$

$\ln 40 - \ln 100 = -\frac{2}{3}\left(-\frac{3}{250}\ln\frac{4}{5}\right)t^{\frac{3}{2}}$

$\ln\frac{2}{5} = \frac{1}{125}\ln\frac{4}{5}t^{\frac{3}{2}}$

$t = 64.1$