Complex Numbers

A. THE GROWTH OF THE NUMBER SYSTEM

In primitive societies all that are needed are the counting numbers, 1, 2, 3, … (or even just the first fewof these).
Greek thinkers gradually came to terms with the existence of such irrational numbers, and by 370 BC Eudoxus had devised a very careful theory of proportion which included both rational and irrational numbers. In AD 876. This was the final element needed to complete the set of real numbers, consisting of positive and negative rational and irrational numbers and zero.
Complex numbers were regarded with great suspicion for many years. Descartes called them 'imaginary', Newton called them 'impossible', and Leibniz's mystification has already been quoted. But complex numbers turned out to be very useful, and had become accepted as an essential tool by the time Gauss first gave them a firm logical basis in 1831.

B. WORKING WITH COMPLEX NUMBERS

Faced with the problem of wanting the square root of a negative number, we make the following Bold Hypothesis.
The real number system can be extended by including a new number, denoted by i, which combines with it self and the real numbers according to the usual laws of algebra, but which has the additional property that i^2 = - 1.
The original notation for i was ι, the Greek letter iota. The letter j is also commonly used instead of i. The first thing to note is that we do not need further symbols for other square roots. For example, since -196 = 196 x (-1) = 14^2 x i^2, we see that - 196 has two square roots, ±14i.
The following example uses this idea to solve a quadratic equation with no real roots.

The general methods for addition, subtraction and multiplication are similarly straightforward.

- **Addition**: add the real parts and add the imaginary parts.

$$(x+iy)+(u+iv)=(x+u)+i(y+v)$$

- **Subtraction**: subtract the real parts and subtract the imaginary parts.

$$(x+iy)-(u+iv)=(x-u)+i(y-v)$$

- **Multiplication**: multiply out the brackets in the usual way and simplify, remembering that i^2 = -1

$$(x+iy)(u+iv)=xu+ixv+iyu+i^2yv$$
$$=(xu-yv)+i(xv+yu)$$

Division of complex numbers is dealt with later in the chapter.

Complex Conjugates

The complex number $x-iy$ is called the *complex conjugate*, or just the *conjugate*, of $x+iy$. Similarly $x+iy$ is the complex conjugate of$x-iy$. $x+iy$ and $x-iy$ are a conjugate pair. The complex conjugate of z is denoted by z*. If a polynomial equation, such as a quadratic, has real coefficients, then any complex roots will be conjugate pairs. You can solve quadratic equations with complex coefficients in the same way as an ordinary quadratic, either by completing the square or by using the quadratic formula.

Example :
Solve $z^2-4iz+13=0$
Solution :
Substitute a = 1, b = -4i and c = -13 into the quadratic formula.

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{4i\pm\sqrt{(-4i)^2-4\times1\times(-13)}}{2}$$
$$=\frac{4i\pm\sqrt{-16+52}}{2}$$
$$=\frac{4i\pm\sqrt{36}}{2}$$
$$=\frac{4i\pm6}{2}$$
$$=2i\pm3$$

So the roots are 3 + 2i and -3 + 2i

Division of Complex Numbers

Two complex numbers z = x + iy and w = u +iv are equal if both x =u and y = v. If u ≠ x or v ≠ y, ot both, then z and w are not equal. The rational numbers $\frac{x}{y}$ and $\frac{u}{v}$ are equal if x = u and y = v. So for two complex numbers to be equal, the real parts must be equal and the imaginary parts must be equal.

The Square Root of a Complex Number

The next example shows you how to find the square root of a complex number.
Example :
Find the two square roots of 8 + 6i.
Solution : Let $(x + iy)^2$ = 8 + 6i → $x^2-2ixy+y^2=8\ +\ 6i$ $\quad\{-y^2=+i^2y^2$
Equating the real and imaginary party gives :

Real : $x^2y^2=8$
Imaginary : 2xy = 6

Rearranging gives : y = $\frac{3}{x}$
Substituting into gives : $x^2-\frac{9}{x^2}$ = 8

$$x^4-9=8x^2$$
$$x^4-8x^2-9=0$$
$$(x^2-9)(x^2+1)=0$$

$x^2 = -1$ which has no real roots

Or $x^2 = 9 \rightarrow x = \pm 3$

When x = 3, y = 1

When x = -3 , y = -1

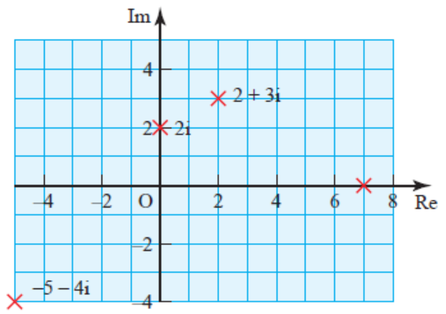
So that square roots of 8 + 6i are 3 + i and -3 –i.

C. REPRESENTING COMPLEX NUMBERS GEOMETRICALLY

Since each complex number $x + iy$ can be defined by the ordered pair of real numbers (x, y), it is natural to represent $x + iy$ by the point with cartesian co-ordinates (x, y).

For example, in figure :

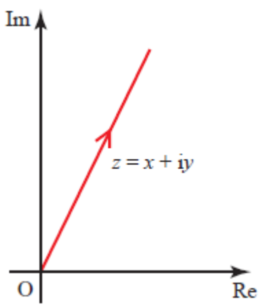
- 2 + 3i is represented by (2, 3)
- 2i is represented by (0, 2)
- 7 is represented by (7, 0)



All real numbers are represented by points on the x axis, which is therefore called the real axis. Pure imaginary numbers (of the form 0 + iy) give points on the y axis, which is called the imaginary axis. It is useful to label these Re and Im respectively. This geometrical illustration of complex numbers is called the complex plane or the Argand diagram.

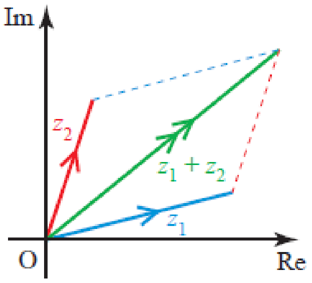
Representing the Sum and Difference of Complex Numbers

- The complex number *x* ⍰ *iy* is represented by the position vector $\binom{x}{y}$ as shown in figure.

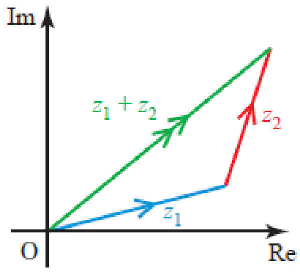


- The addition of complex numbers can then be shown by the addition of the corresponding vectors.

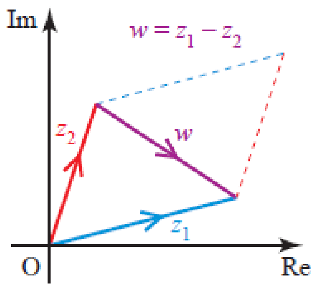
$\binom{x_1}{y_1} + \binom{x_2}{y_2} = \binom{x_1+x_2}{y_1+y_2}$. The position vectors representing z_1 and z_1 form two sides of a parallelogram, the diagonal of which is the vector $z_1 + z_1$.



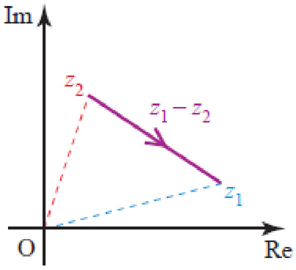
- You can also represent z by any other directed line segment with components $\binom{x}{y}$, not anchored at the origin as a position vector. Then addition can be shown as a triangle of vectors.



- If you draw the other diagonal of the parallelogram, and let it represent the complex number . $z_2 + w = z_1 \rightarrow w = z_1 - z_2$

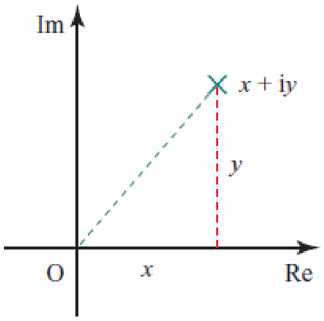


- The complex number $z_1 - z_2$is represented by the vector from the point representing z_2 to the point representing z_1, Notice the order of the points: the vector $z_1 - z_2$starts at the point z_2 and goes to the point z_1.



The Modulus of a Complex Number

Shows the point representing $z = x + iy$ on an Argand diagram.



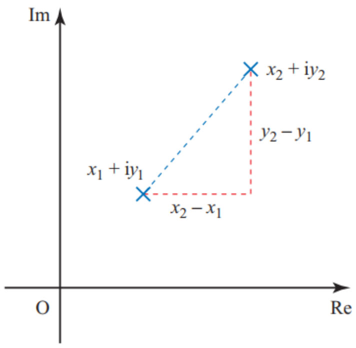
Using Pythagoras' theorem, you can see that the distance of this point from the origin is $\sqrt{x^2+y^2}$. This distance is called the modulus of z, and is denoted by $|z|$. So for the complex number $z = x + iy$, $|z| = \sqrt{x^2+y^2}$ If z is real, z = x say, then $|z| = \sqrt{x^2}$, which is the absolute value of x, i.e. $|x|$. So the use of the modulus sign with complex numbers fits with its previous meaning for real numbers.

D. SETS OF POINTS IN ARGAND DIAGRAM

If $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$, then

$\boldsymbol{z_2 - z_1 = x_2 - x_1 + i(y_2 - y_1)}$.

So, $\boldsymbol{|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$



By using pythagoras' theorem, you can see that the distance between z_1 and z_2 is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

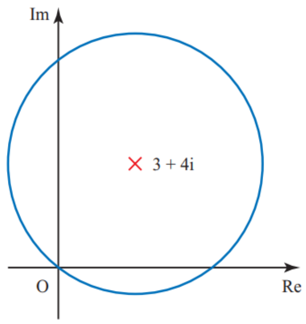
So, $|z_2 - z_1|$ is the distance between the points z_2 and z_1.

Example :

Draw an Argand diagram showing the set of points z for which $|z - 3 - 4i| = 5$.

$|z - 3 - 4i|$ can be written as $|z - (3 + 4i)|$, and this is the distance from the point $3 + 4i$ to the point z.

This equals 5 if the point z lies on the circle with centre $3 + 4i$ and radius 5.

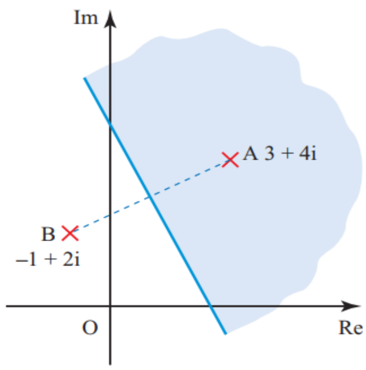


Draw an Argand diagram showing the set of points z for which $|z - 3 - 4i| \leq |z + 1 - 2i|$.

Solution

The condition can be written as $|z - (3 + 4i)| \leq |z - (-1 + 2i)|$.

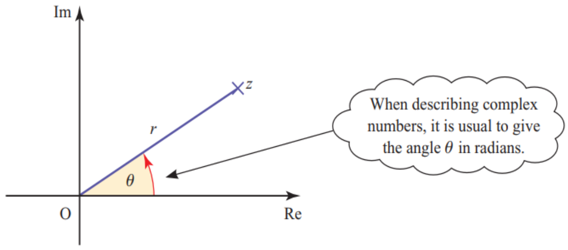
$|z - (3 + 4i)|$ is the distance of point z from the point $3 + 4i$ (point A) and $|z - (-1 + 2i)|$ is the distance of point z from the point $-1 + 2i$ (point B).



These distances are equal if z is on the perpendicular bisector of AB.

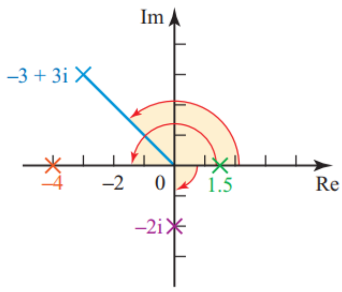
So the given condition holds if z is on this bisector or in the half plane on the side of it containing A.

E. THE MODULUS-ARGUMENT FORM OF COMPLEX NUMBER



The distance r is $|z|$.
The angle θ is slightly more complicated, it is measured anticlockwise from the positive real axis, **normally in radian**. However, it is not uniquely defined since adding any multiple of 2π to θ gives the same direction. To avoid confusion, it is usual to choose that value of θ for which $-\pi < \theta \leq \pi$. This is called the **principal argument of z**. Then every complex number except zero has a unique principal argument. The argument of zero is undefined.

Example :

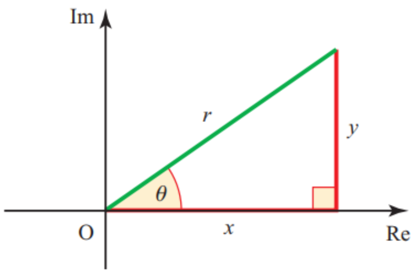


$\text{Arg}(-4) = \pi$

$\text{Arg}(-2i) = -\frac{\pi}{2}$

$\text{Arg}(1.5) = 0$

$\text{Arg}(-3+3i) = \frac{3\pi}{4}$



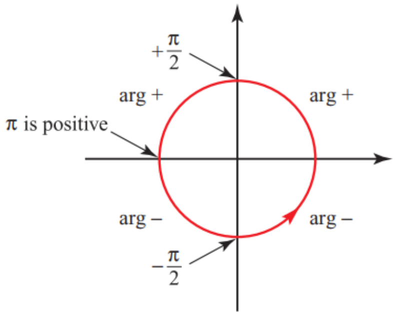
From the picture,

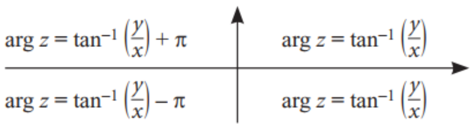
$x = r\ \cos\theta \qquad y = r\sin\theta$

$r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$

Same relations hold in the other quadrants.
Since $x = r\cos\theta$ and $y = r\sin\theta$, we can write the complex number $z = x + iy$ in the form
$z = r(\cos\theta + i\sin\theta)$.

This is called the **modulus-argument** or **polar form**.
You can find the modulus and argument of the complex number by using the relation $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. It is tempting to say that $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
This gives a value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, which is correct only if z is in the first or fourth quadrants.





A complex number in polar form must be given in the form $z = r(\cos\theta + i\sin\theta)$, not in the form $z = r(\cos\theta - i\sin\theta)$. **The value of r must also be positive.**
Example $-2(\cos\alpha + i\sin\alpha)$ is not in a polar form
Using some of the relationships

$\boldsymbol{\cos(\pi-\alpha) = -\cos\alpha}$	$\boldsymbol{\sin(\pi-\alpha) = \sin\alpha}$
$\boldsymbol{\cos(\alpha-\pi) = -\cos\alpha}$	$\boldsymbol{\sin(\alpha-\pi) = -\sin\alpha}$
$\boldsymbol{\cos(-\alpha) = \cos\alpha}$	$\boldsymbol{\sin(-\alpha) = -\sin\alpha}$

You can rewrite the complex number

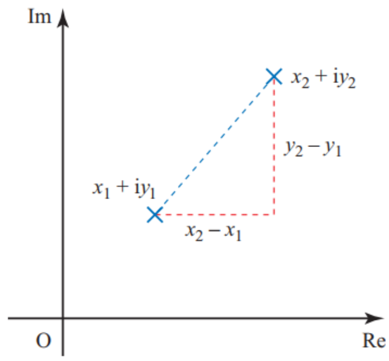
$-2(\cos\alpha + i\sin\alpha) = 2(-\cos\alpha - i\sin\alpha)$

$\qquad = 2(\cos(\alpha-\pi) + i\sin(\alpha-\pi))$.

The modulus is 2 and the argument is $\alpha - \pi$.

F. SETS OF POINTS USING IN POLAR FORM

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then
$z_2 - z_1 = x_2 - x_1 + i(y_2 - y_1)$.
$\arg(z_2 - z_1) = \tan^{-1}\frac{y_2-y_1}{x_2-x_1}$

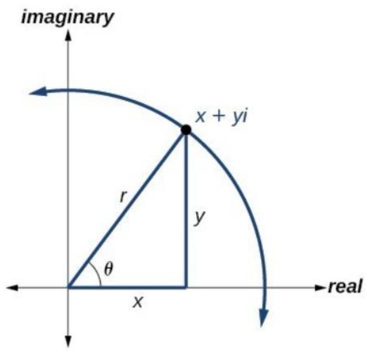


So $\arg(z_2 - z_1)$ is the angle between the line joining z_1 and z_2 and a line parallel to the real axis.

C. WORKING WITH COMPLEX NUMBER IN POLAR FORM

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$



We use the term modulus to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$z = x + yi$$

$$z = r\cos\theta + (r\sin\theta)\,i$$

$$z = r(\cos\theta + i\sin\theta)$$

Products of Complex Number in Polar Form

If there are $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\boldsymbol{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]}$$

This is the complex number with modulus r1r2 and argument (θ1 θ2), so we have that result

$$|z_1 z_2| = |z_1||z_2|$$

And

arg $(z_1 z_2)$ =arg z_1 +arg $z_2(\pm 2\pi$ if necessary, to give the principal arguments)

So to multiply complex numbers in polar form you multiply their moduli and add their arguments.

Quotients of Complex Number in Polar Form

To divide complex numbers in polar form you divide their moduli and subtract their arguments.

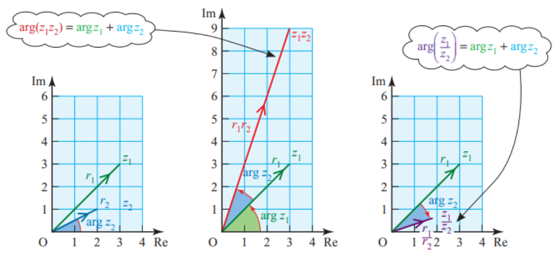
$$\frac{z_1}{z_2} = w$$

Then $z_1 = wz_2$ so that $|z_1| = |w||z_2|$ and arg $z_1 =$ arg $w +$ arg $z_2(\pm 2\pi$ if necessary).

Therefore $|w| = \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and

arg $w =$ arg $\frac{z_1}{z_2}$

$=$ arg $z_1 -$ arg $z_2(\pm 2\pi$ if necessary, to give the principal arguments)



(i) z_1 and z_2 (ii) Multiplying z_1 by z_2 (iii) Dividing z_1 by z_2

To obtain the vector $z_1 z_2$ enlarge the vector z_1 by the scale factor $|z_2|$ and rotate it through arg z_2 anticlockwise about O (figure ii).

To obtain the vector $\frac{z_1}{z_2}$ enlarge the vector z_1 by scale factor $\frac{1}{|z_2|}$ and rotate it clockwise through arg z_2 about O (figure iii).

In summary :

$|z_1 z_2| = |z_1||z_2|$

arg $(z_1 z_2) =$ arg $z_1 +$ arg z_2

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \rightarrow$ arg $\left(\frac{z_1}{z_2}\right) =$ arg $z_1 -$ arg z_2

$$\boldsymbol{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]}$$

Powers of Complex Number in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n.

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\boldsymbol{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$
$$\boldsymbol{z^n = r^n cis(n\theta)}$$

Where n is a positive integer.

Roots of Complex Number in Polar Form

Use the nth root theorem or De Moivre's Theorem and raise the complex number to power with a rational exponent to find the nth root of a complex number in polar form.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right]$$

Where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

H. COMPLEX EXPONENTS

Let $z = \cos\theta + i\sin\theta$. Since i behaves like any other constant in algebraic manipulation, to differentiate z with respect to θ you simply differentiate the real and imaginary parts separately. This gives

$$\begin{aligned}\frac{dz}{d\theta} &= -\sin\theta + i\cos\theta\\ &= i^2\sin\theta + i\cos\theta\\ &= i(\cos\theta + i\sin\theta)\\ &= iz\end{aligned}$$

If i continues to behave like any other constant when it is used as an index, then the general solution of $\frac{dz}{d\theta} = iz$ is $z = e^{i\theta+c}$, where c is a constant, just as $x = e^{kt+c}$ is the general solution of $\frac{dx}{dt} = kx$.

Therefore $\cos\theta + i\sin\theta = e^{i\theta+c}$

Putting $\theta = 0$ gives

$$\begin{aligned}\cos\theta + i\sin\theta &= e^{0+c}\\ 1 &= e^c\\ c &= 0\end{aligned}$$

And it follows that

$$e^{i\theta} = \cos\theta + i\sin\theta\,, \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

The problem with this is that you have no way of knowing how i behaves as an index. But this does not matter. Since no meaning has yet been given to e^z when z is complex.

NOTE :

The particular case when $\theta = \pi$ gives $e^{i\pi} = \cos\pi + i\sin\pi = -1$, so that

$$e^{i\pi} + 1 = 0$$

This statement linking the five fundamental numbers 0, 1, i, e, and π the three fundamental operations of addition, multiplication and exponentiation. The fundamental relation of equality, has been described as a 'mathematical poem'.

I. COMPLEX NUMBERS AND EQUATIONS

The reason for inventing complex numbers was to provide solutions for quadratic equations which have no real roots, such as to solve $az^2 + bz + c = 0$ when the discriminant $b^2 - 4ac$ is negative. If $b^2 - 4ac = -k^2$ (where k is real) then the formula for solving quadratic equations gives $= \frac{-b \pm ik}{2a}$. Notice that when the coefficients of the quadratic equation are real, these roots are a pair of conjugate complex numbers.

All polynomial equations (even those with complex coefficients) can be solved by means of complex numbers. In 1629 Albert Girard, stated that an nth degree polynomial equation has precisely n roots, including complex roots and taking into account repeated roots.

For example, the fifth degree equation $(z-2)(z-4)^2(z^2+9) = 0$ has five roots : 2, 4 (twice), $3i$ and $-3i$.

To find the solution of a particular equation you may be able to use an exact method, such as the formula for the roots of a quadratic equation. There are much more complicated formulae for solving cubic or quartic equations, but not in general for equations of degree five or more.

EXERCISE

1. [9709_s13_qp_31_07a]
   Without using a calculator, solve the equation
   $3w + 2iw^* = 17 + 8i$,
   Where w^* denotes the complex conjugate of w. Give your answer in the form $a + bi$.

   Answer:
   Let $w = a + bi$ and $w^* = a - bi$
   $3(a + bi) + 2i(a - bi) = 17 + 8i$
   $3a + 3bi + 2ai + 2b = 17 + 8i$
   $3a + 2b + 2ai + 3bi = 17 + 8i$
   $3a + 2b = 17$
   $2a + 3b = 8$
   Use elimination method to find a and b.

2. [9709_s14_qp_33_07a]
   The complex number $\frac{3-5i}{1+4i}$ is denoted by u.
   Showing your working, express u in the form $x + iy$, where x and y are real.

   Answer:
   $$\frac{3-5i}{1+4i} \times \frac{1-4i}{1-4i}$$

$$= \frac{3 \times 1 + 3 \times -4i + (-5i) \times 1 + (-5i) \times (-4i)}{1 \times 1 + 4i \times -4i}$$

$$\frac{3 - 20 + (-12i) + (-5i)}{1 + 16} = \frac{-17 - 17i}{17}$$

$$= -1 - i$$

3. [9709_w16_qp_31_09a]

Throughout this question the use of a calculator is not permitted.

Solve the eqution $(1 + 2i)w^2 + 4w - (1 - 2i) = 0$, giving your anwer in the form $x + iy$, where x and y are real.

Answer:

Use formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\frac{-4 \pm \sqrt{4^2 - 4 \times (1 + 2i) \times (-1 + 2i)}}{2(1 + 2i)}$$

$$= \frac{-4 \pm \sqrt{16 - 4(-5)}}{2(1 + 2i)} = \frac{-4 \pm \sqrt{36}}{2(1 + 2i)}$$

$$= \frac{-4 \pm 6}{2(1 + 2i)} - \frac{10}{2(1 + 2i)}$$

$$and \frac{2}{2(1 + 2i)} - \frac{5}{1 + 2i}$$

$$and \frac{1}{1 + 2i} - \frac{5}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} = -1 + 2i$$

$$\frac{1}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} = \frac{1}{5} - \frac{2}{5}i$$