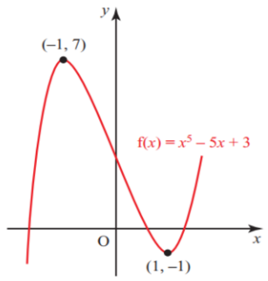
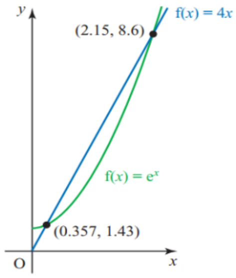
Numerical Solution





- $X^5 - 5x + 3 = 0$ has three roots, lying in the intervals [-2, -1], [0, 1], and [1, -2].
- $e^x = 4x$ has two roots, lying in the intervals $[0,1]$ *and* $[2,3]$.

INTERVAL ESTIMATION – CHANGE-OF-SIGN METHODS

Assume that you are looking for the roots of the equation $f(x) = 0$. This means that you want the values of x for which graph crosses the x-axis, so $f(x)$ changes sign. Provided that $f(x)$ is a continuous function, once you have located an interval number in which $f(x)$ changes sign, that interval must contain a root.

- **Decimal Search**
  In this method, take increments in x of size 0.1 within the interval $[0, 1]$, working out the value of $f(x) = x^5 - 5x + 3$ for each one. Do this until find a change of sign.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
f(x)	3.00	2.50	2.00	1.50	1.01	0.53	0.08	–0.33

There is a sign change in the interval $[0.6, 0.7]$. Now, continue with increments of 0.01 within the interval $[0.6, 0.7]$.

x	0.60	0.61	0.62
f(x)	0.08	0.03	–0.01

This shows that the roots lies in the interval [0.61, 0.62].
Alternative ways of expressing this information are that the root can be taken as 0.615 with a maximum error of ± 0.005, or the root is 0.6 (to 1 decimal places).

This process can be continued by considering $x = 0.611, x = 0.612, \ldots$ to obtain the root to any required number of places.

- **Interval Bisection**
  This method is similar to the decimal search, but instead of dividing each interval into ten parts and looking for a sign change, in this case the interval is divided into two parts – it is bisected.

The root in the interval $[0,1]$, start by taking the mid-point.
$f(0.50) = 0.53$, so $f(0.5) > 0$. Since $f(1) < 0$, the root is in $[0.5,1]$
Take the mid-point of second interval, 0.75.
$f(0.75) = -0.51$, so $f(0.75) < 0$.
Since $f(0.5) > 0$, the root is in $[0.5, 0.75]$.
The mid-point of this further reduced interval is 0.625.
$f(0.625) = -0.03$, so the root is in the interval $[0.5, 0.625]$
The method continues in this manner until any required degree of accuracy is obtained.
However, the interval bisection method is quite slow to coverage to the root and is cumbersome when performed manually.

FIXED-POINT ITERATION

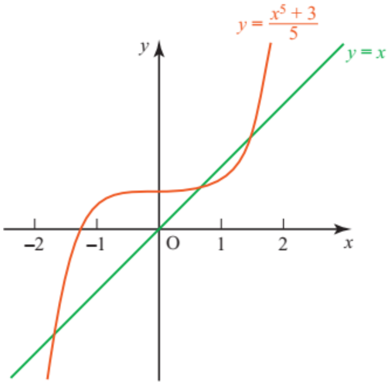
In fixed-point iteration you find a single value or point as your estimate for value of x, rather than establishin an interval.

The iteration process is easiest to understand if you consider the graph. Rewriting the equation $f(x) = 0$ in the form $x = F(x)$ means that instead of looking for points where the graph of $y = f(x)$ crosses the x-axis, you are now finding the points of intersection of the curve $y = f(x)$ and the line $y = x$.

What you do

- Choose a value, x_1, of x
- Find the corresponding value of $F(x_1)$
- Take this value $F(x_1)$ as the new value of x, i.e. $x_2 = F(x_1)$
- Find the value of $F(x_2)$ and so on.

Example



This provides the basis for the iterative formula

$$x_{n+1} = \frac{x_n^5 + 3}{5}$$

Taking $x = 1$ as a starting point to find the root in the interval [0, 1], successive approximation are:

$x_1 = 1,\quad x_2 = 0.8,\quad x_3 = 0.6655,\quad x_4 = 0.6261,$

$x_5 = 0.6192,\quad x_6 = 0.6182,\quad x_7 = 0.6181,$

$x_8 = 0.6180,\quad x_9 = 0.6180$

EXERCISE

1. **[9709_s13_qp_32_02]**

The sequence of values given by the iterative formula

$$x_n = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)}.$$

With initial value $x_1 = 3.5$, converge ti α.

(i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places.

(ii) State an equation satisfied by α and hence find the exact value of α

Answer:

(i)

$$x_n = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)}$$

$x_1 = 3.5$
$x_2 = 3.683702$
$x_3 = 3.684032$
$x_4 = 3.684032$
So, $x = 3.6840$

(ii)

$$x = \frac{x(x^3 + 100)}{2(x^3 + 25)}$$

$2x(x^3 + 25) = x(x^3 + 100)$
$2(x^3 + 25) = (x^3 + 100)$
$2x^3 + 50 = x^3 + 100$
$x^3 = 100 - 50$

$x = \sqrt[3]{50}\ or\ 3.684$

2. **[9709_m16_qp_32_03]**

The equation $x^5 - 3x^3 + x^2 - 4 = 0$ has a positive root.

(i) Verify by calculation that this root lies between 1 and 2.

(ii) Show that the equation can be rearranged in the form

$$x = \sqrt[3]{\left(3x^2 + \frac{4}{x^2} - 1\right)}$$

(iii) Use an iterative formula based on this rearrangement to determine the positive root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

Answer:

(i)

For $x = 1\ \ y = -5$
$x = 2\ \ y = 8$

Because there is a change of sign, so the root is between.

(ii)

$x^5 - 3x^3 + x^2 - 4 = 0$
$x^5 = 3x^3 - x^2 + 4$

Divide by x^2

$$x^3 = 3x + \frac{4}{x^2} - 1$$

$$x = \sqrt[3]{3x + \frac{4}{x^2} - 1}$$

Shown

(iii)

$$x_{n+1} = \sqrt[3]{3x_n + \frac{4}{x_n^2} - 1}$$

Use $x_1 = 1$
$x_2 = 1.8171$
$x_3 = 1.7824$
$x_4 = 1.7765$
$x_5 = 1.7755$
$x_6 = 1.7753$
$x_7 = 1.7753$
So, $x = 1.78$ 2 d.p.

3. [9709_w13_qp_33_05]

It is given that $\int_0^p 4xe^{-\frac{1}{2}x}dx = 9$, where p is a positive constant.

(i) Show that $p = 2\ln\left(\frac{8p+16}{7}\right)$.

(ii) Use an iterative process based on the equation in part (i) to find the value of p correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures.

Answer:

(i) $\int 4xe^{-\frac{1}{2}x}dx$

$u = 4x$

$du = 4\,dx$

$dv = e^{-\frac{1}{2}x}$

$v = -2e^{-\frac{1}{2}x}$

$\int udv = uv - \int vdu$

$\int udv = 4x\left(-2e^{-\frac{1}{2}x}\right) - \int -2e^{-\frac{1}{2}x}4\ dx$

$\int udv = -8xe^{-\frac{1}{2}x} - 16e^{-\frac{1}{2}x} + c$

Put the limit 0 and p and equate to 9

$-8pe^{-\frac{1}{2}p} - 16e^{-\frac{1}{2}p} + 16 = 9$

$-8pe^{-\frac{1}{2}p} - 16e^{-\frac{1}{2}p} = -7$

$e^{-\frac{1}{2}p}(16 + 8p) = -7$

Obtain correct answer

$p = 2\ln\left(\frac{8p + 16}{7}\right)$

(ii) $x_n = 2\ln\frac{8x_{n+1}+16}{7}$

$x_1 = 3.5$

$x_2 = 3.6766$

$x_3 = 3.7398$

$x_4 = 3.7619$

$x_5 = 3.7696$

$x_6 = 3.7723$

$x_5 = 3.7732$

So, $x = 3.77$