Further Algebra

A. THE GENERAL BINOMIAL EXPANSION

Binomial expansion formula, if $n \in \mathbb{N}$.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r$$

This may also be written as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Where If $x \notin \mathbb{N}$, $|x| < 1$.

The coefficients in binomial expansion use the form of a Pascal triangle.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Example :

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

When $n \notin \mathbb{N}$, there are no zeros in the row of binomial coefficients and so we obtain an infinite sequence of non-zero terms. Example :

$n=-3$	gives	1	-3	$\frac{(-3)(-4)}{2!}$	$\frac{(-3)(-4)(-5)}{3!}$	$\frac{(-3)(-4)(-5)(-6)}{4!}$	…
	that is	1	-3	6	-10	15	…

So $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 + \cdots$

NOTE : This expansion are valid only if $|x| < 1$

B. REVIEW OF ALGEBRAIC FRACTIONS

The expression $\left(\frac{f(x)}{g(x)}\right)$ is an algebraic fraction or rational function with $f(x)$ and $g(x)$ are polynomials.

- Simplifying Fractions

  To simplify a fraction (reduce it to lowest terms), the numerator and the denominator must be divided by the same nonzero whole number. A fraction is in lowest terms when the greatest common factor (GCF) of its numerator and denominator is one.

  Example, arithmetic :

  $$\frac{16}{32} = \frac{16 \times 1}{16 \times 2} = \frac{1}{2}$$

  And in algebra

  $$\frac{6a}{9a^2} = \frac{2 \times 3 \times a}{3 \times 3 \times a \times a} = \frac{2}{3a}$$

  Another example :

  $$\frac{2a+4}{a^2-4} = \frac{2(a+2)}{(a+2)(a-2)} = \frac{2}{(a-2)}$$

- Multiplying & Dividing Fractions

  When multiplying fractions, remove any factors that are equal to both the numerator and denominator.

  $$\frac{10a}{3b^2} \times \frac{9ab}{25} = \frac{2 \times 5 \times a}{3 \times b \times b} \times \frac{3 \times 3 \times a \times b}{5 \times 5} = \frac{6a^2}{5b}$$

  In multiplying fractions, it is often necessary to factor in algebraic expressions first such as simplifying.

  When one fraction is divided by another, you change $\div$ to $\times$ and invert the fraction which follows the $\div$ symbol. Example :

  $$\frac{12}{x^2-4} \div \frac{4}{x+2} = \frac{12}{(x+2)(x-2)} \times \frac{x+2}{4}$$

  $$= \frac{3}{x-2}$$

- Addition & Subtraction of Fractions

  To add or subtract two fractions they must be replaced by equivalent fractions, both of which have the same denominator. Example :

  $$\frac{2x}{3} + \frac{x}{4} = \frac{8x}{12} + \frac{3x}{12} = \frac{11x}{12}$$

  be careful when subtracting a fraction causes a sign change.

  When two denominators have no common factor, their product gives the new denominator.

  $$\frac{2}{y+3} + \frac{3}{y-2} = \frac{2(y-2)+3(y+3)}{(y+3)(y-2)}$$

  $$= \frac{2y-4+3y+9}{(y+3)(y-2)}$$

  $$= \frac{5y+5}{(y+3)(y-2)}$$

  $$= \frac{5(y+1)}{(y+3)(y-2)}$$

C. PARTIAL FRACTIONS

When integrating, it is easier to work with a number of simple fractions than a combined one. For example, the only analytic method for integrating $\frac{1}{(1+2x)(1+x)}$ involves first writing it as $\frac{2}{(1+2x)} - \frac{1}{(1+x)}$. This process of taking an expression such as $\frac{1}{(1+2x)(1+x)}$ and writing it in the form $\frac{2}{(1+2x)} - \frac{1}{(1+x)}$ is called expressing the algebraic fraction in partial fractions.

When finding partial fractions you must always assume the most general numerator Possible.

Type 1: Denominators of the form $(ax + b)(cx + d)(ex + f)$

Express $\frac{4+x}{(1+x)(2-x)} = \frac{A}{1+x} - \frac{B}{2-x}$
Multiplying both sides by (1 + x)(2 – x) gives
4 + x = A(2 – x) + B(1 + x).
This is an identity; it is true for all values of x.
There are two possible ways in which you can find the constants A and B. You can either
● substitute any two values of x in (two values are needed to give two
equations to solve for the two unknowns A and B); or
● equate the constant terms to give one equation (this is the same as putting x = 0) and the coefficients of x to give another.

Method 1: Substitution

Although you can substitute any two values of *x*, the easiest to use are x = 2 and
x = -1, since each makes the value of one bracket zero in the identity.

$4 + x = A(2 - x) + B(1 + x)$
$x = 2 \rightarrow 4 + 2 = A(2 - 2) + B(1 + 2)$
$6 = 3B \rightarrow B = 2$
$x = -1 \rightarrow 4 - 1 = A(2 + 1) + B(1 - 1)$
$3 = 3A \rightarrow A = 1$

Substituting these values for *A* and *B* gives $\frac{4+x}{(1+x)(2-x)} = \frac{1}{1+x} - \frac{2}{2-x}$

Method 2: Equating coefficients

In this method, you write the right-hand side of
$4 + x = A(2 - x) + B(1 + x)$
as a polynomial in x, and then compare the coefficients of the various terms.

$4 + x = 2A - Ax + B + Bx$
$4 + x = 2A - Ax + B + Bx$
$4 + 1x = (2A + B) + (-A + B)x$

Equating the constant terms: 4 = 2*A* + *B* These are simultaneous equations in *A* and *B*.
Equating the coefficients of *x*: 1 = ?? + *B*
Solving these simultaneous equations gives *A* = 1 and *B* = 2 as before In some cases it is necessary to factorise the denominator before finding the partial fractions.

Type 2: Denominators of the form $(ax + b)(cx^2 + d)$

Express $\frac{2x+3}{(x-1)(x^2+4)}$ as a sum of partial fractions

SOLUTION

You need to assume a numerator of order 1 for the partial fraction with a
denominator of $x^2 + 4$, which is of order 2.

$\frac{2x+3}{(x-1)(x^2+4)} = \frac{A}{x-1} - \frac{Bx+C}{x^2+4}$ Bx + C is the most general numerator of order 1.

Multiplying both sides by $(x - 1)(x^2 + 4)$gives
$2x + 3 = A(x^2 + 4) + (Bx + C)(x - 1)$
$x = 1$; $5 = 5A \rightarrow A = 1$
The other two unknowns, *B* and *C*, are most easily found by equating coefficients. Identity may be rewritten as :
$2x + 3 = (A + B)x^2 + (-B + C)x + (4A - C)$
Equating coefficients of x^2 : $0 = A + B \rightarrow B = -1$
Equating constant terms : $3 = 4A - C \rightarrow C = 1$
This gives

$$\frac{2x + 3}{(x - 1)(x^2 + 4)} = \frac{1}{x - 1} - \frac{1 - x}{x^2 + 4}$$

Type 3: Denominators of the form $(ax + b)(cx + d)^2$

The factor $(cx + d)^2$ is of order 2, so it would have an order 1 numerator in the
partial fractions. However, in the case of a repeated factor there is a simpler form.

Consider $\frac{4x+5}{(2x+1)^2}$

This can be written as $\frac{2(2x+1)+3}{(2x+1)^2} = \frac{2(2x+1)}{(2x+1)^2} + \frac{3}{(2x+1)^2} = \frac{2}{(2x+1)} + \frac{3}{(2x+1)^2}$

Note
In this form, both the numerators are constant.
In a similar way, any fraction of the form $\frac{px+q}{(cx+d)^2}$ can be written as $\frac{A}{(cx+d)} + \frac{B}{(cx+d)^2}$
When expressing an algebraic fraction in partial fractions, you are aiming to find the simplest partial fractions possible, so you would want the form where the numerators are constant.
Example questions :
Express $\frac{x+1}{(x-1)(x-2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$
Multiplying both sides by $(x - 1)\ (x - 2)^2$ gives
Notice that you only need $(x – 2)^2$ here and not $(x – 2)^3$
$x + 1 = A(x - 2)^2 + B(x - 1)(x - 2) + C(-1)$
X = 1 (so that x -1 =0) → 2 = A$(-1)^2$ → A = 2
X = 2 (so that x – 2 = 0) → 3 = C
Equating coefficients of x^2 → 0 = A + B → B = -2
This gives
$\frac{x+1}{(x-1)(x-2)^2} = \frac{2}{x-1} + \frac{2}{x-2} + \frac{3}{(x-2)^2}$

D. USING PARTIAL FRACTIONS WITH THE BINOMIAL EXPANSION

One of the most common reasons for writing an expression in partial fractions is to enable binomial expansions to be applied, as in the following example.

Express $\frac{2x+7}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$

Multiplying both sides by (x – 1)(x + 2) gives

$2x + 7 = A(x + 2) + B(x - 1)$

X = 1 → 9 = 3A → A = 3

X = -2 → 3 = -3B → B = -1

This gives $\frac{2x+7}{(x-1)(x+2)} = \frac{3}{(x-1)} + \frac{1}{(x+2)}$

In order to obtain the binomial expansion, each bracket must be of the form
(1 ± ...), giving

$\frac{2x+7}{(x-1)(x+2)} = \frac{-3}{(x-1)} + \frac{1}{2(1+\frac{x}{2})}$

$$= -3(1-x)^{-1} - \frac{1}{2}(1+\frac{x}{2})^{-1}$$

The two binomial expansions are

For $|x| < 1$

$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \cdots$

$= 1 + x + x^2$ and for $\left|\frac{x}{2}\right| < 1$

$(1+\frac{x}{2})^{-1} = 1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}(-x)^2 + \cdots$

$= 1 - \frac{x}{2} + \frac{x^2}{4}$

Substituting these in gives

$\frac{2x+7}{(x-1)(x+2)} = -3(1 + x + x^2) - \frac{1}{2}(1 - \frac{x}{2} + \frac{x^2}{4})$

$$= -\frac{7}{2} - \frac{11x}{4} - \frac{25x^2}{8}$$

The expansion is valid when $|x| <$ 1 and $\frac{x}{2}$ 1. The stricter of these is $|x| < 1$

KEY POINTS

1. The general binomial expansion for $n \in R$ *is*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} + \cdots$$

In the special case when $n \in N$, the series expansion is finite and validfor all *x*.
When $n \nexists$, the seriesexpansion is non-terminating(infinite) and valid only if $|x| < 1$.

2. hen $n \nexists$, $(a + x)^n$ should be written as $a^n(1 + \frac{x}{a})^n$ before obtaining the binomial expansion.

3. When multiplying algebraic fractions, you can only cancel when the same factor occurs in both the numerator and the denominator.
4. When adding or subtracting algebraic fractions, you first need to find a common denominator.
5. The easiest way to solve any equation involving fractions is usually to multiply both sides by a quantity which will eliminate the fractions.
6. A proper algebraic fraction with a denominator which factorises can be decomposed into a sum of proper partial fractions.
7. The following forms of partial fraction should be used.

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

$$\frac{px^2+qr+r}{(ax+b)(cx^2+d)} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}$$

$$\frac{px+q}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

E. EXERCISE

1. Show that, for small values of x^2

$$(1-2x^2)^{-2} - (1+6x^2)^{\frac{2}{3}} \approx kx^4$$

Where the value of the constant k is to be determined

Answer :

- Obtain correct(unsimplified) version x^2 or x^4 termn in $(1\text{-}2x^2)^{-2}$
- Obtain $1+ 4x^2$
- Obtain ... + $12x^4$
- Obtain correct(unsimplified) version x^2 or x^4 termn in $(1+6x^2)^{\frac{2}{3}}$
- Obtain $1 + 4x^2 – 4x^4$
- Combine expansions to obtain k = 16 with no error seen

2. Expand $\frac{1}{\sqrt[3]{(1+6x)}}$ in ascending powers of x, up to and including the term x^3, simplifying the coefficients.

Answer :

- State a correct unsimplified version of the x or x^2 or x^3 term in the expansion of $(1+6x)^{-\frac{1}{3}}$
- State correct first two terms 1 -2x
- Obtain term $8x^2$
- Obtain term $-\frac{112}{3}x^3$ $\left(37\frac{1}{3}x^3\right)$ in final answer.

3. Expand $(1+3x)^{-\frac{1}{3}}$ in ascending powers of x, up to and including the term x^3, simplifying the coefficients.

Answer :

- State a correct unsimplified version of the x or x^2 or x^3 term
- State correct first two terms 1 – x
- Obtain the next two terms $2x^2 - \frac{14}{3}x^3$
- (Symbolic binomial coefficients, e.g. $\binom{-\frac{1}{3}}{3}$ are not sufficient for the M mark.)