Binomial Distribution

A. THE BINOMIAL DISTRIBUTION

In Binomial Distribution, we emphasise the fact by using the word **trial** rather than the more general term **experiment.**

- You are conducting **trials** on random samples of a certain size, denoted by **n.**
- These are just **two possible outcomes.** These are often referred to as **success** and **failure.**
- Both outcomes have fixed probabilities, the **two adding to 1.** The probability of **success** is usually called p, that of **failure** q, so $p + q = 1$.

Possible values of X and theor probabilities are shown in the table below.

r	0	1	2	3
$P(X = r)$	q^3	$3pq^2$	$3p^2q$	p^3

This values of X with their associated **probabilities** is called a **binomial probability distribution,** a special case of a discrete random variable.

r	0	1	2	3	4	5
$P(X = r)$	q^5	$5pq^4$	$10p^2q^3$	$10p^3q^2$	$5p^4q$	p^5

10 is ${}^5C_2$.

The entry for $X = 2$, for example, there are **two success,** giving probability p^2 and **three failures,** giving probability q^3. This can be written as
$P(X = 2) = 10p^2q^3$

If you familiar with binomial theorem, you will notice that the probabilities in the table are the terms of binomial expansion of $(q + p)^5$. This is why called **binomial distribution.** Sum of these probabilities is $(q + p)^5 = 1^5$, since $\boldsymbol{q + p = 1}$, which is to be expected since the distribution covers all the possible outcomes.

The general case

The general binomial distribution deals with the possible numbers of successes when there are $\boldsymbol{n}$ trials, each of which may be a success **(with probability p),** or a failure **(with probability q)**; p and q are fixed positive numbers and $p + q = 1$. This distribution is denoted by $B(n, p)$.

For $B(n, p)$, the probability of r successes in n trial is found by the same argument as before. Each success has probability p and failure q, so the probability of r success and $(n - r)$ failures in a particular order is $p^r q^{n-r}$. The position in the sequence of n trials which the successes occupy can be chosen C_r^n ways.

Therefore,

$P(X = r) = C_r^n p^r q^{n-r}$ for $0 \le r \le n$.

This can also be written as

$$P_r = \binom{n}{r} p^r (1-p)^{n-r}$$

The successive probabilities for $X = 0,1,2, \dots, n$ are the terms of the binomial ecpansion of $(q + p)^n$

Note:

$X \sim B(n, p)$ symbol $\sim$ means 'has the distribution'

B. THE EXPECTATION AND VARIANCE OF $\boldsymbol{B\ (n,\ p)}$

- **Expectation**

If $X \sim B(n, p)$,
then the expectation or mean of $X = \mu = np$

- **Variance**, Var $(X) = \sigma^2 = npq = np(1-p)$
- **Standard deviation** $= \sigma = \sqrt{npq} = \sqrt{np(1-p)}$

C. EXERCISE

1. **[9709_w15_qp_61_001]**

In a certain town. 76% of cars are fitted with satellite navigation equipment. A random sample of 11 cars from this town is chosen. Find the probability that fewer than 10 of these cars are fitted with this equipment.

Answer:
$p = 0.76$
P (fewer than 10) $= 1 - P(10,11)$
$= 1 - (C_{10}^{11}(0.76)^{11}(0.24) + (0.76)^{11})$
$= 1 - 0.219$
$= 0.781$

2. **[9709_s13_qp_62_004]**
Robert uses his calculator to generate 5 random integers between 1 and 9 inclusive.
   (i) Find the probability that at least 2 of the 5 integers are less than or equal to 4.

Robert now generates n random integers between 1 and 9 incusive. The random variable X is the number of these n integers which are less than or equal to a certain integer k between 1 and 9 inclusive. It is given that the mean of X is 96 and the variance of X is 32.
   (ii) Find the values of n and k.

Answer:

(i) $p = \frac{4}{9}$

P(at least 2) $= 1 - P(0,1)$

$= 1 - \left(\left(\frac{5}{9}\right)^5 - C_1^5 \left(\frac{4}{9}\right)\left(\frac{5}{9}\right)^4\right)$

$= 0.735$

(ii) $np = 96, npq = 32$

$q = \frac{npq}{np} = \frac{32}{96} = \frac{1}{3}$

$p = 1 - q$

$p = 1 - \frac{1}{3} = \frac{2}{3}$

$96 = np$

$n = 144$

$p = \frac{2}{3} = \frac{k}{9}$

$k = 6$

3. **[9709_s15_qp_62_001]**

A fair die is thrown 10 times. Find the probability that the number of sixes obtained is between 3 and 5 inclusive.

Answer:

P for six in dice $= \frac{1}{6}$

$P(3,4,5) =$

$C_3^{10}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7 + C_4^{10}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6 + C_5^{10}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^5$

$= 0.222$