Permutation & Combination

A. FACTORIAL

Factorial is the product of all positive integers less than or equal to a given positive integer and denoted by that integer and an exclamation point. In general the number of ways of placing *n* different objects in a line is *n!* , where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ with *n* must be a positive integer.

Examples :

- Calculate $\frac{8! \times 6!}{3! \times 4!}$

  Solution

  $$\frac{8! \times 6!}{3! \times 4!} = \frac{8 \times 7 \times 6 \times 5 \times 4! \times 6 \times 5 \times 4 \times 3!}{3! \times 4!}$$
  $$= 8 \times 7 \times 6 \times 5 \times 6 \times 5 \times 4$$
  $$= 201600$$

- Find the number of ways in which all six letters in the word REVIEW can be arranged.

  Solution

  There are $6! = 720$ arrangements of five letters. However, REVIEW has two repeated letters and so some of these arrangements are really the same.

  R E V I E W

  The two E's can be arranged in $2! = 2$ ways, so the total number of arrangements is

  $$\frac{6!}{2!} = \frac{720}{2} = 360$$

B. PERMUTATION

In general the number of permutations,

$$P(n,r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$$

This can be written more compactly as

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Permutations with repetition :

$$n^r$$

Permutations without repetitions :

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Examples :

- If you have six snacks and you want to send them to four of your friends, how many ways can this be done?

  Solution

  We have to find number of permutations of four objects out of six objects. Hence we know, $r = 4$ and $n = 6$

  As repetitions are not allowed here, we will use the formula :

  $$^{n}P_{r} = \frac{n!}{(n-r)!}$$

  $$^{6}P_{4} = \frac{6!}{(6-4)!}$$

  $$^{6}P_{4} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!}$$

  $$^{6}P_{4} = 6 \times 5 \times 4 \times 3$$

  $$^{6}P_{4} = 360$$

C. COMBINATION

The combination is defined as ways in which an element from a set is selected, such that (unlike permutations) the order of selection does not matter.

To work out combinations, given formulas are :

Number of combinations of r objects chosen from n objects when no repetition is allowed is defined as :

$$C_r^n = \frac{n!}{r!\,(n-r)!}$$

Number of combination of r objects chosen from n objects when repetition is allowed is defined as :

$$\frac{(n+r-1)!}{r!\,(n-1)!}$$

Examples :

- A School Governors committee of five people is to be chosen from eight applicants. How many different selections are possible?

  Solution

  Number of selections $= C_5^8 = \frac{8!}{5! \times 3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$

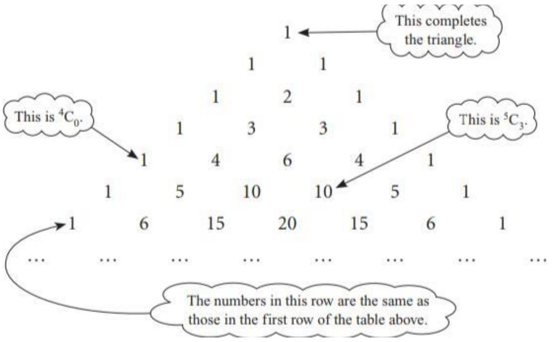
D. THE BINOMIAL COEFFICIENTS

In the last section you met numbers of the form ^{n}C, or $\binom{n}{r}$. These are called the bionomial coefficients, the reasin for this is explained in appendix 3 (which you cand find on the CD) and in the next chapter.

Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ And the results $\binom{n}{0} = \binom{n}{n} = 1$ to check that the entries in this table, for n=6 and n=7, are correct.

r	0	1	2	3	4	5	6	7
n = 6	1	6	15	20	15	6	1	-
n = 7	1	7	21	35	35	21	7	1

It is very common to present values of ^nC, in a table shaped like an isosceles triangle, known as Pascal's triangle.



Symmetry : $^nC_r = {}^nC_{n-r}$

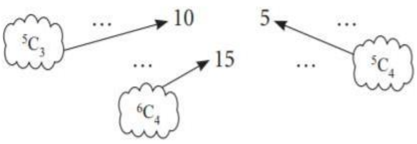
This Provides a short cut in calculations when r is large. For example

$^{100}C_{96} = {}^{100}C_4 = \frac{100\times 99\times 98\times 97}{1\times 2\times 3\times 4} = 3921225$

It also shows that the list of value of nCr for any particular value of n is unchanged by being reversed. For example, when n=6 the list the seven numbers 1, 2 ,3 , 4, 5, 6, 1

Addition : $^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$

Look at the entry 15 in the bottom row of Pascal's triangle, towards the right. The two entries above and either side of it are 10 and 5.



And 15 = 10 +5. In this case $^6C_4 = {}^5C_3 = {}^5C_4$. This is an example of the general result that $^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$

Using Biomial Coefficients to calculate probabilities

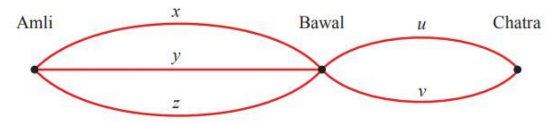
Example :

A committee of 5 is to be chosen from a list of 14 people, 6 of whom are men and 8 women. Their names are to be put in a hat and then 5 drawn out. What is the probability that this procedure produces a committee with no women

Solution :

The probability of an all-male committee of 5 people is given by

There are three roads from Amli to Bawal and two roads from Bawal to Chatra. How many routes are there from Amli to Chatra passing through Bawal on the way?



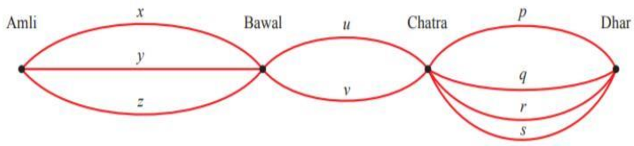
The answer is 3 × 2 = 6 because there are three ways of doing the first leg, followed by two for the second leg.

The six routes are :

$x-u \quad y-u \quad z-u$

$x-y \quad y-v \quad z-v$

There are also four roads from Chatra to Dhar. So each of the six routes from Amli to Chatral has four possible ways of going on to Dhar. There are now 6 × 4 = 24 routes.



They can be listed systematically as follows :

$x-u-p$	$y-u-p$	$z-u-p$	$x-v-p$	$y-v-p$	$z-v-p$
$x-u-q$	… … … .	… … … …	… … … …	… … …	… … …
$x-u-r$	… … …	… … …	… … … …	… … … .	… … …
$x-u-s$	… … … .	… … …	… … … …	… … …	… … …

Example :

A cricket team consisting of 6 batsmen, 4 bowlers and 1 wicket-keeper is to be selected from a group of 18 cricketers comprising 9 batsmen, 7 bowlers and 2 wicket-keepers. How many different teams can be selected?

Solution :

The batsmen can be selected in $^9C_6$ ways.

The blowlers can be selected in $^7C_4$ ways.

The wicket-keepers can be selected in $^2C_1$ ways.

Therefore total numbers of teams

$= {}^9C_6 \times {}^7C_4 \times {}^2C_1$

$$= \frac{9!}{3!\times 6!}\times\frac{7!}{3!\times 4!}\times\frac{2!}{1!\times 1!}$$

$$= \frac{9\times 8\times 7}{3\times 2\times 1}\times\frac{7\times 6\times 5}{3\times 2\times 1}\times 2$$

$= 5880$

E. KEY POINTS

1. The number of ways of arranging n unlike objects in a line in n!
2. N! = n x (n - 1) x (n – 2) x (n – 3) x ... x 3 x 2 x 1.
3. The number of distinct arrangements of n objects in a line, of which p are identical to each other, q others are identical to each other, r of a third type are identical, and so on is $$\frac{n!}{p!\,q!\,r!\,\ldots}$$
4. The number of permutations of r objects from n is $${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$$
5. The number of combinations of r objects from n is $${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

F. EXAMPLE

1. In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels must always come together?

Explanation :

The word 'MATHEMATHICS' has 11 letters. It has the vowels 'A' 'E' 'A' 'I' in it and these 4 vowels must always come together. Hence these 4 vowels can be grouped and considered as a single letter. That is MTHMTCS(AEAI).

Hence we can assume total letters as 8. But in these 8 letters 'M' occurs 2 times, 'T' occurs 2 times but rest of the letters are different.

Hence number of ways to arrange these letters

$$=\frac{8!}{(2!)(2!)} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{(2\times1)(2\times1)} = 10080$$

In the 4 vowels (AEAI) 'A' occurs 2 times and rest of the vowels are different.

Number of ways to arrange these vowels among themeselves $= \frac{4!}{2!} = \frac{4\times3\times2\times1}{2\times1} = 12$

Hence required number of ways = 10080 x 12 =120960

2. In how many can a group of 5 men and 2 women be made out a total 7 men and 3 women?
Explanation:

We need to select 5 men from 7 men and 2 women from 3 women.

Number of ways to do this $[\because {}^{n}C_{r} = {}^{n}C_{(n-r)}]$

$= {}^{7}C_{5} \times {}^{3}C_{2}$

$= {}^{7}C_{2} \times {}^{3}C_{1}$

$= \frac{7\times6}{2\times1} \times 3 = 21\times3 = 63$

3. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?
Explanation :
Number of ways of selecting 3 consonants from 7
$= {}^{7}C_{3}$
Number of ways of selecting 2 vowels from 4
$= {}^{4}C_{2}$
Number of ways of selecting 3 consonants from 7 and 2 vowels from 4
$= {}^{7}C_{3} \times {}^{4}C_{2}$
$= \left(\frac{7\times6\times5}{3\times2\times1} \times \frac{4\times3}{2\times1}\right)$
$= 210$

It means we can have 210 groups where each group contains total 5 letters *(3 consonants and 2 vowels)*.

Number of ways of arranging 5 letters among themselves
$= 5! = 5\times4\times3\times2\times1 = 120$
Hence, required number of ways
$= 210\times120 = 25200$

4. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there in the committee. In how many ways can it be done?

Explanations :

From a group of 7 men and 6 women, five persons are to be selected with at least 3 men.
Hence we have the following 3 options.

We can select 5 men ...(option 1)
Number of ways to do this = ${}^{7}C_{5}$
We can select 4 men and 1 woman ...(option 2)
Number of ways to do this = ${}^{7}C_{4} \times {}^{6}C_{1}$

We can select 3 men and 2 women ...(option 3)
Number of ways to do this = ${}^{7}C_{3} \times {}^{6}C_{2}$

Total number of ways
$= {}^{7}C_{5} + ({}^{7}C_{4} \times {}^{6}C_{1}) + ({}^{7}C_{3} \times {}^{6}C_{2})$

$= {}^7C_2 + ({}^7C_3 \times {}^6C_1) + ({}^7C_3 \times {}^6C_2)$

$[\because {}^nC_r = {}^nC_{(n-r)}]$

$$= \frac{7 \times 6}{2 \times 1} + \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 6 + \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{6 \times 5}{2 \times 1}$$

=21+210+525

=756

5. Ten different letters of alphabet are given. Words with six letters are formed from these given letters. Find the number of words which have at least one letter repeated?

Explanation :

Initially let's find out the number of six letter words which can be formed from ten different letters when any letter may be repeated any number of times.

Any of the 1010 letters can be placed at each place of the 6-letter word.

10	10	10	10	10	10

Hence, number of 6-letter words which can be formed from ten different letters when any letter may be repeated any number of times (this will also include the number of words formed when no letter is repeated) =106=106 ...(A)

Number of 6-letter words which can be formed from ten different letters when no letter is repeated = ${}^{10}P_6$ ...(B)

from (A) and (B)
required number of ways
=106 =106 - ${}^{10}P_6$